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# Contact classification of linear ordinary differential equations. II.

by

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**Contact classification of linear ordinary differential equations. II.**

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ABSTRACT. The classification of ODEs $y^{(n)} = \sum_{i=2}^{n} a_{n-i}(x)y^{(n-i)}$, $n \geq 3$, in a neighborhood of a regular point up to a contact transformation is given.

## 1. INTRODUCTION

This paper is devoted to the local classification of linear ODEs of order $n \geq 3$ in neighborhoods of regular points up to a contact transformation.

In [2], E.Cartan proved that for $n \leq 2$, any $n$-order linear ODE can be transformed to the form $y^{(n)} = 0$ by a point transformation. For $n \geq 3$, it is incorrect. In this case, there are infinite number of different equivalence classes of $n$-order linear ODEs.

First the problem of local classification of linear ordinary differential equation (ODE) up to a transformation of variables was set up by classics of XIX century E. Laguerre, G.-H. Halphen and others. They obtained first results concerning the classification of linear ODEs of 3-rd and 4-th orders, see [7], [3]. Essentially, this problem was forgotten after them.

It is well known that any linear ODE can be transformed by a point transformation to the form

$$y^{(n)} = a_{n-2}(x)\,y^{(n-2)} + a_{n-3}(x)\,y^{(n-3)} + \ldots + a_0(x)\,y\,. \tag{1.1}$$

In this paper, we classify linear ODEs of this form. The case $n = 3$, we considered in [13], [14].

In his book [12], E.J.Wilczynski proved that any linear ODE of order $n \geq 3$ can be transformed by a point transformation to the form

$$y^{(n)} = a_{n-3}(x)\,y^{(n-3)} + a_{n-4}(x)\,y^{(n-4)} + \ldots + a_0(x)\,y \tag{1.2}$$

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(see also, [10],[4]). In [15], [16], we obtained the local classification of linear ODEs of form (1.2).

Our approach to the problem is the following.

In their paper [8], F. M. Mahomed and P. G. L. Leach proved that dimension of the algebra of point symmetries of a $n$-order linear ODE is equal to either $n+4$ or $n+2$, or $n+1$.

We prove (theorem 2.2) that dimension of algebra of point symmetries of a linear ODE is an invariant of contact transformations that take the set of linear ODEs to itself.

Further, we prove (theorem 4.1) that any linear ODE with $n+4$-dimensional algebra of point symmetries is reduced by a point transformation to the form $y^{(n)} = 0$. For linear ODEs with $n+2$ and $n+1$-dimensional algebras of point symmetries, we prove (theorem 2.3) that a contact transformation that takes one of these equation to another one is a point transformation. These results reduce the problem of local classification of linear ODEs w.r.t. contact transformations to the classification w.r.t. point transformations.

Further, we reduce (theorem 2.5) the problem to the classification w.r.t. point transformations of the form

$$X = f(x), \quad Y = |\, f'(x)\,|^{(n-1)/2} y \tag{1.3}$$

In his thesis [9], F. M. Mahomed proved that any linear ODE with $n+2$-dimensional algebra of point symmetries can be transformed to a linear ODE with constant coefficients by a point transformation of form (1.3). We find invariants of transformations of linear ODEs with constant coefficients and $n+2$-dimensional algebras of point symmetries. These invariants solve the equivalence problem for these equations (theorem 4.8). Finally, we classify linear ODEs with constant coefficients up to equivalence (theorem 4.11).

Further, we calculate (theorem 4.17) the algebra of scalar differential invariants of linear ODEs. We use these invariants to classify linear ODEs with $n+1$-dimensional algebras of point symmetries. Let $\mathcal{E}$ be an arbitrary linear ODE with $n+1$-dimensional algebra of point symmetries and let $I(x)$ be its nonconstant scalar differential invariant. The transformation of form (1.3) $X = I(x)$, $Y = |\, I'(x)\,|^{(n-1)/2} y$ takes $\mathcal{E}$ to the equation $\mathcal{E}'$. We say that $\mathcal{E}'$ is a canonical form of $\mathcal{E}$. We prove (theorem 4.23) that equivalent equations have the same canonical form. It leads to the classification (theorem 4.24) of linear ODEs with $n+1$-dimensional algebras of point symmetries in a neighborhood of regular point up to equivalence.

Below, all manifolds and maps are supposed to be smooth. By $\mathbb{R}^n$ denote the $n$-dimensional arithmetical space.

## 2. Preliminaries

In this section, we recall necessary notations and results of the geometry of differential equations ([5], [6]) and linear ODEs ([8],[12]). We prove some necessary results concerning symmetries and transformations of linear ODEs too.

### 2.1. **Jet bundles.**

2.1.1. *Cartan distribution.* Let $E$ and $M$ be a smooth manifolds of dimensions $n+m$ and $n$ respectively and let $\pi : E \to M$ be a smooth bundle. By $[S]_x^k$ denote the $k$–jet of a section $S$ of $\pi$ at the point $x \in M$. By

$$\pi_k : J^k\pi \to M\,,\ \pi_k : [S]_x^k \mapsto x\,,\ k = 0, 1, 2, \ldots, \infty\,,$$

denote the bundle of $k$–jets of all sections of $\pi$.

The projection $\pi_{k,r} : J^k\pi \to J^r\pi\,,\ k > r$, is defined by $\pi_{k,r}(\,[S]_x^k\,) = [S]_x^r$.

Every section $S$ of $\pi$ generates the section $j_kS$ of the bundle $\pi_k$ by the formula $j_kS : x \mapsto [S]_x^k$. By $L_S^{(k)}$ denote the image of the section $j_kS$.

By $T_{x_k}(J^k\pi)$ denote the tangent space to $J^k\pi$ at $x_k \in J^k\pi$, by $T_{x_k}(L_S^{(k)})$ denote the tangent space to $L_S^{(k)}$ at $x_k \in L_S^{(k)}$.

Let $x_k \in J^k\pi$. Consider all submanifolds $L_S^{(k)}$ passing trough $x_k$ and consider their tangent spaces $T_{x_k}(L_S^{(k)})$ at $x_k$. The subspace $C_{x_k} \subset T_{x_k}(J^k\pi)$ spanned on the union of these tangent spaces is called the *Cartan plane at* $x_k$. The distribution $C : x_k \mapsto C_{x_k}$ is called the *Cartan distribution on* $J^k\pi$.

2.1.2. *Lie transformations.* A (local) diffeomorphism of $J^k\pi$ that takes the Cartan distribution to itself is called a *Lie transformation.* A Lie transformation of $J^0\pi$ (that is an arbitrary diffeomorphism of $J^0\pi$) is called a *point transformation.* A Lie transformation of $J^1\pi$ is called a *contact transformation* if $m = 1$.

Every Lie transformation $f : U \to U'$ of $J^k\pi$ can be lifted canonically to the Lie transformation $f^{(r)} : \pi_{k+r,k}^{-1}(U) \to \pi_{k+r,k}^{-1}(U')$ of $J^{k+r}\pi$, $r = 1, 2, \ldots$, such that, for $r \geq l$, the diagram

$$\begin{array}{ccc}
\pi_{k+r,k}^{-1}(U) & \xrightarrow{f^{(r)}} & \pi_{k+r,k}^{-1}(U') \\
{\scriptstyle \pi_{k+r,k+l}}\downarrow & & \downarrow{\scriptstyle \pi_{k+r,k+l}} \\
\pi_{k+l,k}^{-1}(U) & \xrightarrow[f^{(l)}]{} & \pi_{k+l,k}^{-1}(U')
\end{array}$$

is commutative. Indeed, $f^{(r)}$ is defined in the following way. A point $x_{k+1} = [S]_x^{k+1} \in J^{k+1}\pi$ is identified with $K_{x_{k+1}} = T_{x_k}(L_S^{(k)})$, where $x_k = \pi_{k+1,k}(x_{k+1})$. The differential $f_*$ maps $K_{x_{k+1}}$ onto the subspace $f_*(\,K_{x_{k+1}}\,)$. If $f_*(\,K_{x_{k+1}}\,)$ is projected on $M$ without a degeneration, then there is $x'_{k+1} \in J^{k+1}\pi$ such that $K_{x'_{k+1}} = f_*(K_{x_{k+1}})$ and we set $f^{(1)}(x_{k+1}) = x'_{k+1}$. It is obvious that $f^{(1)}$ is a Lie transformation of $J^{k+1}\pi$ defined almost everywhere in $\pi^{-1}_{k+1,k}(U)$. Setting $f^{(r+1)} = (f^{(r)})^{(1)}$, we define the Lie transformation $f^{(r)}$ for all $r = 1, 2, \ldots$. Clearly, that $f^{(r)}$ is defined almost everywhere in $\pi^{-1}_{k+r,k}(U)$.

It is well known (see [5], [6]) that any Lie transformation is the lifting of some point transformation if $m > 1$ and if $m = 1$, then any Lie transformation is the lifting of some contact transformation.

2.1.3. *Lie fields.* A vector field $\xi$ in $J^k\pi$ is called a *Lie field* if its flow is generated by Lie transformations. A vector field in $J^0\pi$ is said to be a *point vector field.* A Lie field in $J^1\pi$ is called a *contact vector field* if $m = 1$.

The canonical lifting of Lie transformations generates the lifting of Lie fields. Indeed, let $\xi$ be a Lie field in $J^k\pi$ and let $f_t$ be its flow. Then the flow $f_t^{(r)}$ of the Lie transformations defines the Lie field $\xi^{(r)}$ in $J^{k+r}\pi$ , $r = 1, 2, \ldots$ such that

$$(\pi_{k+r,k+l})_*\xi^{(r)} = \xi^{(l)}\,, \quad r \geq l\,.$$

## 2.2. **Ordinary differential equations.**

2.2.1. *Contact classification.* Let $\pi : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ and let $x,\ y,\ p_1,\ \ldots, p_k$ be the standard coordinates on $J^k\pi$.

Any $k$-order ODE

$$F(x, y(x), dy/dx, \ldots, d^ky/dx^k) = 0$$

is identified with the submanifold

$$\mathcal{E} = \{\ F(x, y, p_1, \ldots, p_k) = 0\ \} \subset J^k\pi\,.$$

An "usual" solution $S(x)$ of the initial ODE is identified with the submanifold $L_S^{(k)} \subset \mathcal{E}$ generated by the section $S : x \mapsto S(x)$ of $\pi$. Obviously, $L_S^{(k)}$ is a 1-dimensional integral manifold of the Cartan distribution on $J^k\pi$. A "many-valued" solution of $\mathcal{E}$ is a 1-dimensional integral manifold $L$ of the Cartan distribution on $J^k\pi$ such that $L \subset \mathcal{E}$. It is easy to prove that locally, almost everywhere, a "many-valued" solution has the form $L_S^{(k)}$.

It is natural to classify $k$-order ODEs up to a diffeomorphism of $J^k\pi$ that takes the set of all solutions of ODEs to itself. Obviously, this diffeomorphism is a Lie transformation. Hence, it is generated by a contact transformation. So, we come to the problem of classification of ODEs up to a contact transformation.

Let $\mathcal{E}_1$, $\mathcal{E}_2 \subset J^k\pi$ be $k$-order ODEs and $f$ be a point (contact) transformation. We say that $f$ takes (locally) $\mathcal{E}_1$ to $\mathcal{E}_2$ if $f^{(k)}$ $(f^{(k-1)})$ takes (locally) the submanifold $\mathcal{E}_1 \subset J^k\pi$ to the submanifold $\mathcal{E}_2 \subset J^k\pi$. We say that ODEs $\mathcal{E}_1$ and $\mathcal{E}_2$ are *equivalent* if there exists a point (contact) transformation that takes (locally) $\mathcal{E}_1$ to $\mathcal{E}_2$.

2.2.2. *Point and contact transformations.* Any point transformation $f$ is defined in the standard coordinates by the formulae

$$X = X(x,y)\,, \quad Y = Y(x,y)\,. \tag{2.1}$$

Obviously, the lifting $f^{(k)}$ is defined in the standard coordinates by the formulae

$$X = X(x,y)\,,\ Y = Y(x,y)\,,\ P_1 = \frac{DY}{DX}\,,\ \ldots,\ P_k = \frac{DP_{k-1}}{DX}\,, \tag{2.2}$$

where

$$D = \frac{\partial}{\partial x} + p_1\frac{\partial}{\partial y} + p_2\frac{\partial}{\partial p_1} + \ldots + p_{k+1}\frac{\partial}{\partial p_k} + \ldots \tag{2.3}$$

is the operator of total derivation w.r.t. $x$.

It is easy to show that a contact transformation is defined in the standard coordinates by the formulae

$$X = X(x,y,p_1)\,, \quad Y = Y(x,y,p_1)\,, \quad P_1 = \frac{Y_x + p_1Y_y}{X_x + p_1X_y}\,, \tag{2.4}$$

where functions $X(x,y,p_1), Y(x,y,p_1)$ are connected by the relation

$$Y_{p_1}(X_x + p_1X_y) - X_{p_1}(Y_x + p_1Y_y) = 0\,.$$

2.2.3. *Point and contact vector fields.* Let $\xi$ be a contact vector field. It is easy to prove that $\xi$ can be represented in the standard coordinates as

$$\xi = \xi_\varphi = -\varphi_{p_1}\frac{\partial}{\partial x} + (\varphi - p_1\varphi_{p_1})\frac{\partial}{\partial y} + (\varphi_x + p_1\varphi_y)\frac{\partial}{\partial p_1}\,, \tag{2.5}$$

where the function $\varphi = \varphi(x,y,p_1)$ is called the *generating function* of $\xi$.

Let $\zeta$ be an arbitrary point vector field. It has the form

$$\zeta = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} \tag{2.6}$$

in the standard coordinates. The lifting $\zeta^{(1)}$ is a contact vector field. It is easy to verify that the generating function of $\zeta^{(1)}$ is

$$b(x,y) - a(x,y)\cdot p_1\,. \tag{2.7}$$

Conversely, if the generating function of a contact vector field has form (2.7), then this vector field is the lifting of some point vector field. We shall say that function (2.7) is the *generating function of point vector field* (2.6).

Let us transform a contact vector field by an arbitrary contact transformation (2.4). As a result, we obtain a new contact vector field. It is easy to verify that the generating functions $\Phi$ of the obtained vector field and the generating function $\varphi$ of the initial one are connected by the formula

$$\frac{X_x + p_1X_y}{X_xY_y - X_yY_x}\,\Phi(\,X,\ Y,\ P_1\,) = \varphi(x,y,p_1)\,. \tag{2.8}$$

2.2.4. *Classical symmetries.* A point vector field $\zeta$ in $J^0\pi$ is called a *point symmetry* of a differential equation $\mathcal{E} \subset J^n\pi$ if $\zeta^{(n)}$ is tangent to the submanifold $\mathcal{E}$. By $\operatorname{Pnt}\mathcal{E}$ we denote the set of all point symmetries of $\mathcal{E}$.

A contact vector field $\xi$ in $J^1\pi$ is called a *contact symmetry* of a differential equation $\mathcal{E} \subset J^n\pi$ if $\xi^{(n-1)}$ is tangent to the submanifold $\mathcal{E}$ and there is no a point vector field $\zeta$ with $\xi = \zeta^{(1)}$.

The point and contact symmetries are called *classical symmetries.* By $\operatorname{Sym}\mathcal{E}$ we denote the set of all classical symmetries of $\mathcal{E}$.

The space of generating functions of classical symmetries of an ODE

$$p_n - F(x,\, y,\, p_1, \dots, p_{n-1}) = 0$$

coincide with the space of solutions of the form $\varphi = \varphi(x, y, p_1)$ of the linear PDE (see [5], [6])

$$\Big(\, \bar{D}^n - \frac{\partial F}{\partial p_{n-1}}\bar{D}^{n-1} - \frac{\partial F}{\partial p_{n-2}}\bar{D}^{n-2} - \ldots - \frac{\partial F}{\partial p_1}\bar{D}^1 - \frac{\partial F}{\partial y}\,\Big)(\varphi) = 0\,, \quad (2.9)$$

where $\bar{D} = \partial/\partial x + p_1\partial/\partial y + p_2\partial/\partial p_1 + \ldots + p_{n-1}\partial/\partial p_{n-2} + F\partial/\partial p_{n-1}$.

## 2.3. **Linear ordinary differential equations.**

2.3.1. *Classical symmetries.* Let $\mathcal{E}$ be an arbitrary $n$-order linear ODE.

In [13],[14], we proved for $n = 3$ the following results: $\dim \operatorname{Sym}\mathcal{E}$ can be equal to either 10 or 5, or 4; if $\dim \operatorname{Sym}\mathcal{E} = 10$, then $\operatorname{Sym}\mathcal{E}$ is generated by three contact and seven point symmetries; if $\dim \operatorname{Sym}\mathcal{E} = 5$ or 4, then $\operatorname{Sym}\mathcal{E} = \operatorname{Pnt}\mathcal{E}$.

**Proposition 2.1.** *If* $n > 3$*, then* $\operatorname{Sym}\mathcal{E} = \operatorname{Pnt}\mathcal{E}$.

*Proof.* We can assume without loss of generality that $\mathcal{E}$ has form (1.1).

Let $\varphi(x, y, p_1)$ be the generating function of an arbitrary classical symmetry of $\mathcal{E}$. The generating function of a point symmetry has the form $\alpha(x, y)p_1 + \beta(x, y)$. Hence we should be check that $\varphi_{y^{(1)}y^{(1)}} \equiv 0$ to prove the theorem. To this end, let us consider equation (2.9) for $\mathcal{E}$:

$$(\, \bar{D}^n - a_{n-2}\bar{D}^{n-2} - a_{n-3}\bar{D}^{n-3} - \ldots - a_1\bar{D}^1 - a_0\,)\varphi = 0\,. \qquad (2.10)$$

Obviously,

$$\begin{aligned}
\bar{D}^n(\varphi) &= \bar{D}^{n-2}(\, \varphi_{p_1}p_3 + \varphi_{p_1p_1}(p_2)^2 + \text{low degree terms}\,) = \\
&= \begin{cases} \bar{D}(\, 3\varphi_{p_1p_1}p_2p_3 + \text{l. d. t.}\,), & \text{if } n = 4 \\ \bar{D}^{n-3}(\, \varphi_{p_1}p_4 + 3\varphi_{p_1p_1}p_2p_3 + \text{l. d. t.}\,), & \text{if } n > 4 \end{cases} \\
&= \begin{cases} 3\varphi_{p_1p_1}(p_3)^2 + \text{l. d. t.}, & \text{if } n = 4 \\ (\binom{n-3}{n-5} + 3\binom{n-3}{n-4} + 3)\varphi_{p_1p_1}p_3p_{n-1} + \text{l. d. t.}, & \text{if } n > 4 \end{cases}
\end{aligned}$$

It now follows from (2.10) that $\varphi_{p_1p_1} \equiv 0$. $\square$

Obviously, dimension of an algebra of classical symmetries is an invariant of contact transformations of ODEs. From above mentioned results of [13], [14] and this proposition, we get

**Theorem 2.2.** *Dimension of the algebra of point symmetries of a linear ODE is an invariant of contact transformations that take the set of all linear ODEs to itself.*

2.3.2. *Point symmetries.* Let $\mathcal{E}$ be an arbitrary ODE (1.1). In their paper [8], F. M. Mahomed and P. G. L. Leach proved that a point symmetry of $\mathcal{E}$ has the form

$$(\,\varphi(x)\frac{\partial}{\partial x}+\frac{n-1}{2}\,\varphi'\,y\frac{\partial}{\partial y}\,)+C\,y\frac{\partial}{\partial y}+\gamma(x)\frac{\partial}{\partial y}\,,$$

where $\gamma(x)$ is a solution of $\mathcal{E}$, $C\in\mathbb{R}$, and $\varphi(x)$ is a solution of the system of ODEs

$$\begin{cases} R_n^3\varphi^{(3)}-2a_{n-2}\varphi^{(1)}-a_{n-2}^{(1)}\varphi=0\\ R_n^{k+1}\varphi^{(k+1)}-\displaystyle\sum_{s=0}^{k-3}R_{n-2-s}^{k-1-s}a_{n-2-s}\varphi^{(k-1-s)}\\ \qquad\qquad -ka_{n-k}\varphi^{(1)}-a_{n-k}^{(1)}\varphi=0\,,\ k=3,4\ldots,n\,,\end{cases}\tag{2.11}$$

where $R_p^q=\binom{p}{q-1}(n-1)/2-\binom{p}{q}$. Dimension of the space of solutions of system (2.11) can be equal to either 3 or 1, or 0. It follows that $\dim\operatorname{Pnt}\mathcal{E}$ can be equal to either $n+4$ or $n+2$, or $n+1$.

Thus the set of all linear ODEs $\mathcal{E}$ is divided into the three nonintersecting families according to $\dim\operatorname{Pnt}\mathcal{E}$. From theorem 2.2, we have that these families are invariant w.r.t. contact transformations.

2.3.3. *Contact transformations.* Linear ODEs of order $n\geq 3$ with $n+4$-dimensional algebras of point symmetries have contact symmetries if $n=3$ ( [13], [14] ). It follows that there exist contact transformations of 3-order linear ODEs with 7-dimensional algebras of point symmetries. For $n=3$, the following theorem is proved in [14].

**Theorem 2.3.** *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be linear ODEs with $n+2$ or $n+1$ – dimensional algebras of point symmetries and let $f$ be a contact transformation that takes $\mathcal{E}_1$ to $\mathcal{E}_2$. Then $f$ is the lifting of a point transformation.*

*Proof.* We can assume without loss of generality that $\mathcal{E}_1$ and $\mathcal{E}_2$ have form (1.1).

Suppose $\dim\operatorname{Pnt}\mathcal{E}_1=n+2$. The contact transformation $f$ is defined in the standard coordinates by (2.4). Let $\Gamma_1(X)$, $\Gamma_2(X)$, $\Gamma_3(X)$ be linear independent solutions of $\mathcal{E}_1$. We can consider these solutions as generating functions of point symmetry of $\mathcal{E}_1$ (see subsection 2.3.2). Every that function is connected with the correspondence generating function of point symmetries of $\mathcal{E}_2$ by formula (2.8). Taking into account the form of generating functions of the $n+2$ – dimensional algebra $\mathrm{Pnt}\,\mathcal{E}_2$ (see subsection 2.3.2), we obtain:

$$\Delta\Gamma_1(X(x,y,p_1)) = K_1(\ \varphi(x)p_1 - \frac{n-1}{2}\varphi' y\ ) + C_1 y + \gamma_1(x)$$
$$\Delta\Gamma_2(X(x,y,p_1)) = K_2(\ \varphi(x)p_1 - \frac{n-1}{2}\varphi' y\ ) + C_2 y + \gamma_2(x)$$
$$\Delta\Gamma_3(X(x,y,p_1)) = K_3(\ \varphi(x)p_1 - \frac{n-1}{2}\varphi' y\ ) + C_3 y + \gamma_3(x)$$

where $\Delta = (\ X_x + p_1 X_y\ )/(\ X_x Y_y - X_y Y_x\ )$; $K_j, C_j \in \mathbb{R}$, $j = 1,2,3$. If one of the numbers $K_1$, $K_2$, $K_3$ is not equal to zero, say $K_1 \neq 0$, then

$$\Delta(\ \Gamma_2 - \frac{K_2}{K_1}\Gamma_1\ ) = (\ C_2 - \frac{K_2}{K_1}C_1\ )y + \gamma_2 - \frac{K_2}{K_1}\gamma_1$$
$$\Delta(\ \Gamma_3 - \frac{K_3}{K_1}\Gamma_1\ ) = (\ C_3 - \frac{K_3}{K_1}C_1\ )y + \gamma_3 - \frac{K_2}{K_1}\gamma_1$$

It follows that $(\ K_1\Gamma_2 - K_2\Gamma_1\ )/(\ K_1\Gamma_3 - K_3\Gamma_1\ )$ is independent on $p_1$. Therefore

$$\frac{\partial}{\partial p_1}\left(\frac{K_1\Gamma_2 - K_2\Gamma_1}{K_1\Gamma_3 - K_3\Gamma_1}\right) = \frac{d}{dX}\left(\frac{K_1\Gamma_2 - K_2\Gamma_1}{K_1\Gamma_3 - K_3\Gamma_1}\right) X_{p_1} = 0\,.$$

Suppose that $X_{p_1} \neq 0$, then

$$K_1\Gamma_2 - K_2\Gamma_1 = K(\ K_1\Gamma_3 - K_3\Gamma_1\ ),$$

where $K \in \mathbb{R}$. This means that the solutions $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ of $\mathcal{E}_1$ are linear dependent. From this contradiction, we have $X_{p_1} \equiv 0$. Hence $f$ is the lifting of some point transformation.

Obviously, the proofs for the case $K_1 = K_2 = K_3 = 0$ and for the case $\dim \mathrm{Pnt}\,\mathcal{E} = n+1$ are analogous. □

2.3.4. *Reduction to transformations* (1.3). It is well known that any linear ODE can be reduced to the form (1.1) by a point transformation.

The following proposition holds ([12]).

**Proposition 2.4.** *Let $\mathcal{E}$ be an ODE of form* (1.1). *Then a point transformation takes $\mathcal{E}$ to an ODE of the same form iff this transformation has the form*

$$X = X(x), \quad Y = C \cdot |\,X'(x)\,|^{(n-1)/2} \cdot y + \beta(x), \quad C \in \mathbb{R}\,. \qquad (2.12)$$

It is follows from (2.2) that the lifting of point transformation (2.12) to the Lie transformation of $J^n\pi$ is defined by

$$X = X(x)\,,\ Y = C\,|X'|^{(n-1)/2} y + \beta(x)\,,$$
$$P_k = C\,\nabla^k(\,|X'|^{(n-1)/2} y\,) + \nabla^k(\beta)\,,\ k = 1,2,\ldots,n\,,$$

where $\nabla = D/DX$ and $D$ is operator (2.3).

Let $\mathcal{E}_1 = \{ P_n = A_{n-2}(X)\, P_{n-2} + A_{n-3}(X)\, P_{n-3} + \ldots + A_0(X)\, Y \}$ and $\mathcal{E}_2 = \{ p_n = a_{n-2}(x)\, p_{n-2} + a_{n-3}(x)\, p_{n-3} + \ldots + a_0(x)\, y \}$. Suppose transformation (2.12) takes $\mathcal{E}_1$ to $\mathcal{E}_2$. This means

$$C\, \nabla^n( |X'|^{(n-1)/2} y ) + \nabla^n( \beta(x) ) =$$
$$= \sum_{i=2}^{n} \Big[ C\, A_{n-i}(X(x)) \nabla^{n-i}( |X'|^{(n-1)/2} y ) + A_{n-i}(X(x)) \nabla^{n-i}( \beta(x) ) \Big] .$$

It follows:

(1) the function $\beta(x)$ is a solution of the homogeneous linear ODE $\nabla^n(\beta(x)) = \sum\limits_{i=2}^{n} A_{n-i}(X(x)) \nabla^{n-i}(\beta(x))$.

(2) Transformation (2.12) takes $\mathcal{E}_1$ to $\mathcal{E}_2$ for an arbitrary nonzero constant $C \in \mathbb{R}$.

As a result, we obtain the following statement.

**Theorem 2.5.** *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be ODEs of form* (1.1)*. Then if there exist a point transformation that takes $\mathcal{E}_1$ to $\mathcal{E}_2$, then there exist a point transformation of the form* (1.3) *that takes $\mathcal{E}_1$ to $\mathcal{E}_2$.*

From proposition (2.4), we have that transformations of form (1.3) take the set of all ODEs of form (1.1) to itself.

2.3.5. *Laguerre-Forsyth transformations.* The following proposition holds ([12], [10], [4]).

**Proposition 2.6.** *Let $\mathcal{E} = \{ P_n = A_{n-2}(X)\, P_{n-2} + A_{n-3}(X)\, P_{n-3} + \ldots + A_0(X)\, Y \}$ be an ODE of form* (1.1)*. A point transformation*

$$X = f(x), \quad Y = |\, f' \,|^{(n-1)/2} y \,,$$

*where $f$ is a solution of the ODE*

$$2f'''f' - 3(f'')^2 - 24(\, (n+1)n(n-1)\, )^{-1}\, (f')^4 A_{n-2}(f) = 0 \,,$$

*takes $\mathcal{E}$ to an equation $\mathcal{E}'$ of form* (1.2)*.*

This transformation is called a *Laguerre-Forsyth transformation of the equation $\mathcal{E}$*. The equation $\mathcal{E}'$ is called a *Laguerre-Forsyth form of the equation $\mathcal{E}$*.

It follows from this proposition that the problem of local classification of linear ODEs is reduced to classification of ODEs of form (1.2).

2.3.6. *Reduction to transformations* (2.13)*.* The following proposition holds (see [12]).

**Proposition 2.7.** *Let $\mathcal{E}$ be an ODE of form* (1.2)*. Then a point transformation takes $\mathcal{E}$ to an ODE of the same form iff this transformation has the form*

$$X = \frac{\alpha \cdot x + \beta}{\gamma \cdot x + \delta}\,, \quad Y = C \cdot |\, X' \,|^{(n-1)/2} \cdot y\,, \quad \alpha, \beta, \gamma, \delta, C \in \mathbb{R}\ .$$

Taking into account proposition 2.7, we can prove the following statement by the same way as theorem 2.5.

**Theorem 2.8.** *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be ODEs of form* (1.2). *Then if there exist a point transformation that takes $\mathcal{E}_1$ to $\mathcal{E}_2$, then there exist a point transformation of the form*

$$f(x) = \frac{\alpha\, x + \beta}{\gamma\, x + \delta}\,, \quad \hat{f}(x,y) = |\, f'\,|^{(n-1)/2} \cdot y\,, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}\,, \quad (2.13)$$

*that takes $\mathcal{E}_1$ to $\mathcal{E}_2$.*

From proposition (2.7), we have that transformations of form (2.13) take the set of all ODEs of form (1.2) to itself.

## 3. Bundles of linear ODEs

### 3.1. **Bundle of linear ODEs of form** (1.1).

3.1.1. Here, we reduce the problem of local classification of linear ODEs of form (1.1) to the classification of germs of section of the bundle of these ODEs.

Let $\pi : E = \mathbb{R}^1 \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^1$ be a product bundle. By $x$ denote the standard coordinate on the base $\mathbb{R}^1$, by $a_{n-2}, a_{n-4}, \ldots, a_0$ denote the standard coordinates on the fiber $\mathbb{R}^{n-1}$.

We identify any section

$$S : x \mapsto (\, x, a_{n-2}(x), a_{n-3}(x), \ldots, a_0(x)\,)$$

of $\pi$ with the linear ODE

$$\mathcal{E}_S = \{\, p_n = a_{n-2}(x)\, p_{n-2} + a_{n-3}(x)\, p_{n-3} + \ldots + a_0(x)\, y\,\}\,.$$

Clearly, this identification is a bijection from the set of all sections of $\pi$ to the set of all linear ODEs of form (1.1). By $S_{\mathcal{E}}$ we denote the section of $\pi$ corresponding to the equation $\mathcal{E}$ under this identification.

By $\Gamma$ we denote the Lie pseudogroup of all local diffeomorphisms of $\mathbb{R}^1$. By $\Phi$ we denote the Lie pseudogroup of all point transformations of form (1.3). Formula (1.3) defines the isomorphism

$$\Gamma \to \Phi\,, \quad f \mapsto (\, f, \hat{f}\,)\,, \quad (3.1)$$

where $\hat{f}(x,y) = |\, f'(x)\,|^{(n-1)/2} y$.

Let

$$\mathcal{E}_2 = \{\, P_n = A_{n-2}(X)\, P_{n-2} + A_{n-3}(X)\, P_{n-3} + \ldots + A_0(X)\, Y\,\}$$

be an arbitrary ODE of form (1.1). Subjecting $\mathcal{E}_2$ to an arbitrary transformation $(\, f\,, \hat{f}\,)$ of form (2.13), we obtain linear ODE

$$\mathcal{E}_1 = \{\, p_n = a_{n-2}(x)\, p_{n-2} + a_{n-3}(x)\, p_{n-3} + \ldots + a_0(x)\, y\,\}\,.$$

The coefficients of $\mathcal{E}_1$ are expressed in terms of the coefficients of $\mathcal{E}_2$ and the transformation $f$ by equations of the following form

$$a_{n-j} = F_{n-j}(\, A_{n-2}, \dots , A_{n-j}\, ; \frac{d\,f}{dx}, \dots , \frac{d^{j+1} f}{dx^{j+1}}\,)\,, \ j = 2, 3, \dots , n\,.$$

Obviously, the coefficients of $\mathcal{E}_2$ are expressed in terms of the coefficients of $\mathcal{E}_1$ and the transformation $f^{-1}$ by the same equations

$$A_{n-j} = F_{n-j}(\, a_{n-2}, \dots , a_{n-j}\, ; \frac{d\,f^{-1}}{dX}, \dots , \frac{d^{j+1} f^{-1}}{dX^{j+1}}\,) \qquad (3.2)$$
$$j = 2, 3, \dots , n\,.$$

Equations (3.2) define the lifting of any $f \in \Gamma$ to the diffeomorphism $f^{(0)}$ of the bundle $\pi$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{f^{(0)}} & E \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathbb{R} & \xrightarrow[f]{} & \mathbb{R} \end{array}$$

is commutative (in the domain of $f^{(0)}$).

For any $f \in \Gamma$, we define the transformation of sections of $\pi$ by the formula

$$S \mapsto f(\, S\,) = f^{(0)} \circ S \circ f^{-1}\,. \qquad (3.3)$$

Now equations (3.2) can be represented as

$$S_{\mathcal{E}_2} = f(\, S_{\mathcal{E}_1}\,)\,.$$

Obviously, the following statement holds.

**Proposition 3.1.** *Let* $\mathcal{E}_1\,,\ \mathcal{E}_2$ *be equations of form* (1.1). *Then a transformation* $(\, f\,, \hat{f}\,) \in \Phi$ *takes* $\mathcal{E}_1$ *to* $\mathcal{E}_2$ *iff* $S_{\mathcal{E}_2} = f(\, S_{\mathcal{E}_1}\,)$.

Let $S$ be a section of $\pi$ and let $p$ be a point from the domain of $S$. By $\{S\}_p$ we denote the germ of $S$ at the point $p$. We say that *germs* $\{S_1\}_{p_1}$ *and* $\{S_2\}_{p_2}$ *are equivalent* if there exist $f \in \Gamma$ with

$$f(\, \{S_1\}_{p_1}\,) \stackrel{def}{=} \{f(S_1)\}_{f(p_1)} = \{S_2\}_{p_2}\,.$$

3.1.2. *Symmetries of sections.* Let $S$ be a section of $\pi$ and let $\xi$ be a vector field in the base $\mathbb{R}^1$ of $\pi$. By $f_t$ we denote the flow of $\xi$. We say that $\xi$ is a *symmetry of* $S$ if one of the following equivalent conditions is fulfilled:

(1) the vector field $\xi^{(0)}$ is tangent to the image $L_S^{(0)}$ of $S$;
(2) $\forall t\ f_t(\, S\,) = S$;
(3) $df_t(\, S\,)/dt = 0$.

By $\operatorname{Sym} S$ we denote the Lie algebra of all symmetries of the section $S$.

**Proposition 3.2.** *Let* $S : x \to (\, x, a_{n-2}(x), \dots , a_0(x)\,)$ *be a section of* $\pi$ *and let* $\xi = \varphi(x)\partial/\partial x$. *Then:*

(1) $\xi$ *is a symmetry of* $S$ *iff* $\varphi(x)$ *is a solution of system* (2.11)*;*

(2) $\varphi(x)\partial/\partial x$ *is a symmetry of* $S$ *iff* $\varphi(x)\partial/\partial x+((n-1)/2)\varphi'\,y\,\partial/\partial y$ *is a point symmetry of the equation* $\mathcal{E}_S$ *;*
(3) $\dim \operatorname{Sym} S$ *is equal to either* 3 *or* 1*, or* 0 *;*
(4) $\dim \operatorname{Pnt} \mathcal{E}_S = \dim \operatorname{Sym} S_{\mathcal{E}} + n + 1$ .

*Proof.* All statements of the proposition follows immediately from proposition 3.1 and the results of work [8] of F. M. Mahomed and P. G. L. Leach mentioned in section 2.3.2. □

The following statement is needed for sequel

**Proposition 3.3.** *Let* $\xi$ *be a symmetry of a section* $S$ *of* $\pi$ *and let* $f \in \Gamma$*. Then* $f_*(\xi)$ *is a symmetry of the section* $f(S)$*.*

*Proof.* Let $f_t$ be the flow of $\xi$. Then $f \circ f_t \circ f^{-1}$ is the flow of $f_*\xi$ and $(f \circ f_t \circ f^{-1})(\,f(S)\,) = (f \circ f_t)(S) = f(S)$. □

3.2. **Laguerre-Forsyth bundles.** Let $E^{\,n-3}$ be the subspace of the total space $E$ of $\pi$ defined by

$$E^{\,n-3} = \{\ (\,x,\ a_{n-2}, a_{n-3}, \ldots, a_0\,) \in E \mid a_{n-2} = 0\ \}\,.$$

Then $\tau = \pi|_{E^{\,n-3}} : E^{\,n-3} \to \mathbb{R}$ is a subbundle of the bundle $\pi$. Obviously, the set of sections of $\tau$ is identified with the set of linear ODEs of form (1.2).

By $G$ denote the Lie group of all projective transformations of $\mathbb{R}^1$, that is

$$G \,=\, \{\, f(x) = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \ \mid\ \alpha, \beta, \gamma, \delta \in \mathbb{R} \text{ and } \det \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \neq 0\,\}\,.$$

It is easy to check that the set of nonconstant solutions of the equation

$$2\, f'''\, f' \;-\; 3\,(f'')^2 = 0 \tag{3.4}$$

coincides with $G$.

From proposition (2.7), theorem (2.7), and proposition (3.1), we have that, for any $f \in G$, the diffeomorphism $f^{(0)}$ takes the bundle $\tau$ to itself.

By $\mathfrak{g}$ we denote the Lie algebra of $G$. It can easily be checked that $\mathfrak{g}$ as a vector space over $\mathbb{R}$ is generated by the vector fields

$$\xi_0 = \frac{\partial}{\partial x}\,, \quad \xi_1 = x\frac{\partial}{\partial x}\,, \quad \xi_2 = x^2\frac{\partial}{\partial x}\,. \tag{3.5}$$

It now is obvious that any symmetry of a section of $\tau$ is an element of $\mathfrak{g}$.

Let $S$ be a section of $\pi$ and let $f \in \Gamma$. We say that a transformation $f$ is a *Lagguere-Forsyth transformation for* $S$ if the transformation $(f, \hat{f}) \in \Phi$ is a Lagguere-Forsyth transformation for the ODE $\mathcal{E}_S$.

## 4. Classification of linear ODEs

### 4.1. Classification of linear ODEs with $n+4$ − dimensional algebra of point symmetries.

**Theorem 4.1.** *Let $S$ be a section of $\pi$ with* $\dim \operatorname{Sym} S = 3$ *and let* $\mathbf{0}$ *be the zero section of $\pi$. Then for any $p$ from the domain of $S$, the germs $\{S\}_p$ and $\{\mathbf{0}\}_0$ are equivalent.*

*Proof.* Let $S$ be a section of $\pi$ with $\dim \operatorname{Sym} S = 3$ and let $p$ be a point from the domain of $S$.

Let us choose a Laguerre-Forsyth transformation $f \in \Gamma$ for $S$ such that $f(p) = 0$ ( proposition 2.6 ). Let $S' = f(S) : x \to (\, x, a_{n-3}(x), \ldots, a_0(x)\,)$. From $\dim \operatorname{Sym} S = 3$, we have $\dim \operatorname{Sym} S' = 3$. From proposition 3.2, we have that $\operatorname{Sym} S'$ as a vector space over $\mathbb{R}$ is generated by linear independent vector fields $\varphi_i(x)\partial/\partial x$ , $i = 1, 2, 3$, where functions $\varphi_i$ are solutions of the system (2.11). In the considered case, this system has the form

$$\begin{cases} \varphi''' = 0 \\ 3\, a_{n-3}\varphi' + a'_{n-3}\varphi = 0 \\ \dfrac{(k-1)(n-(k-1))}{2}\, a_{n-k+1}\varphi'' + k a_{n-k}\varphi' + a'_{n-k}\varphi = 0\,, \\ k = 4, 5, \ldots, n\,. \end{cases} \tag{4.1}$$

From the second equation of this system, we have $a_{n-3} \equiv 0$; from the third equation, we have $a_{n-4} \equiv 0$ and so on; from the last equation, we obtain $a_0 \equiv 0$. □

**Corollary 4.2.**

1. *Let $S_1$ and $S_2$ be sections of $\pi$ with 3-dimensional algebras of symmetries. Then any their germs $\{S_1\}_{p_1}$ and $\{S_2\}_{p_2}$ are equivalent.*
2. *The zero section of the bundle $\tau$ is a unique section of $\tau$ that has 3-dimensional algebra of symmetries.*
3. *The zero section of $\tau$ is invariant w.r.t. the group $G$.*

4.2. **Reductions of the classification problem.** From theorem 2.3, proposition 3.1, and theorem 4.1, we have that the problem of local classification of linear ODEs w.r.t. contact transformations is reduced to the problem of classification of section germs of the bundle $\pi$ w.r.t. the pseudogroup $\Gamma$.

Taking into account that $\Gamma$ is a transitive pseudogroup, we obtain that the last problem is reduced to the classification of germs at $0 \in \mathbb{R}^1$ of sections of $\pi$ up to a diffeomorphism from the isotropy group $\Gamma_0 = \{\, f \in \Gamma \mid f(0) = 0\,\} \subset \Gamma$ of the point $0 \in \mathbb{R}^1$.

From proposition 2.6 and theorem 2.8, we now have that the problem of classification of section germs of the bundle $\pi$ w.r.t. the pseudogroup

$\Gamma$ can be reduced to the problem of classification of section germs of the bundle $\tau$ w.r.t. the group $G$. A solution of the last problem is given in [15], [16].

4.3. **Classification of linear ODEs with $n+2$ − dimensional algebra of point symmetries.**

4.3.1. *Linear ODEs with constant coefficients.* Let $S$ be a section of the bundle $\pi$ with $\dim \mathrm{Sym}\, S = 1$. The algebra $\mathrm{Sym}\, S$ as a vector space over $\mathbb{R}$ is spanned on the vector field $\varphi(x)\partial/\partial x$, where $\varphi(x)$ is a unique solution of (2.11).

Let $p$ be a point from the domain of $S$. We say that $p$ is a *regular point of $S$ ( $\mathcal{E}_S$ )* if $\varphi(p) \neq 0$. We say that a germ of $S$ at a regular point is a *regular germ.*

**Theorem 4.3.** *Let $S$ be a section of $\pi$ with* $\dim \mathrm{Sym}\, S = 1$ *and let $\{S\}_p$ be a regular germ. Then there exist a constant section $S'$ of $\pi$ such that $\{S\}_p$ and $\{S'\}_0$ are equivalent.*

*Proof.* In some neighborhood of $p$, there exist a diffeomorphism $f$ of $\mathbb{R}^1$ straightening the vector field $\varphi(x)\partial/\partial x$ (that is the diffeomorphism $f : x \mapsto X$ takes $\varphi(x)\partial/\partial x$ to $\partial/\partial X$) and $f(p) = 0$. It follows that the section $f(S)$ has the symmetry $\partial/\partial X$. Hence, $f(S)$ is a constant section in a neighborhood of $0 \in \mathbb{R}^1$. By $S'$ we denote the constant section of $\pi$ coinciding with $f(S)$ in some neighborhood of $0 \in \mathbb{R}^1$. □

**Corollary 4.4.** *Let $\mathcal{E}$ be a linear ODE of form* (1.1) *with* $\dim \mathrm{Pnt}\, \mathcal{E} = n+2$ *and $p$ is a regular point for $\mathcal{E}$. Then, in some neighborhood of $p$, $\mathcal{E}$ is equivalent to an ODE of form* (1.1) *with constant coefficients.*

Note, that apparently first this statement was proved by F.M.Mahomed in [9].

From theorem 4.3, we have that the problem of classification of sections of $\pi$ with 1-dimensional algebras of point symmetries in a neighborhood of a regular points is reduced to the classification of constant sections of $\pi$ up to equivalence.

Note that any constant section $S$ of $\pi$ has the symmetry of the form $\dfrac{\partial}{\partial x}$. It follows that $\dim \mathrm{Sym}\, S$ is equal to either 1 or 3.

The following proposition makes possible to separate all constant sections with 3 – dimensional algebras of symmetries.

**Proposition 4.5.** *Let $S : x \to ( x, a_{n-2}, \ldots, a_0 )$ be a constant section of $\pi$. Then* $\dim \mathrm{Sym}\, S = 3$ *iff the components $a_{n-k}$ satisfy to the following conditions:*

(1) *if $k$ is odd, then $a_{n-k} = 0$,*

(2) *if $k$ is even, then the coefficients $a_{n-k}$ are expressed in terms of $a_{n-2}$ by the recurrence formula*

$$a_{n-2m} = \frac{1}{2m}\Big( R_n^{2m-1}\lambda^{2m} - \sum_{s=0}^{2m-4} R_{n-2-2s}^{2m-1-2s} a_{n-2-2s}\lambda^{2m-2-2s} \Big)\,,$$

*where* $R_p^q = \binom{p}{q-1}(n-1)/2 - \binom{p}{q}$, $\lambda^2 = \dfrac{2}{R_n^3}\, a_{n-2}$, $m = 2, 4, \ldots$
*In particular, if $a_{n-2} = 0$, then all coefficients are equal to* $0$.

*Proof.* Let $S : x \to (\, x, a_{n-2}, \ldots, a_0\,)$ be a constant section of $\pi$. Then $\dim \mathrm{Sym}\, S = 3$ iff system (2.11) for $S$ has three linear independent solutions. This system has the following form in this case:

$$\begin{cases} R_n^3\varphi^{(3)} - 2a_{n-2}\varphi^{(1)} = 0 \\ R_n^{k+1}\varphi^{(k+1)} - \displaystyle\sum_{s=0}^{k-3} R_{n-2-s}^{k-1-s} a_{n-2-s}\varphi^{(k-1-s)} - ka_{n-k}\varphi^{(1)} = 0\,, \\ k = 3, 4, \ldots, n\,. \end{cases}$$

From the first equation, we obtain that the system has three linear independent solutions: $1, e^{\lambda x}, e^{-\lambda x}$, where $\lambda^2 = (2/R_n^3)a_{n-2}$. It now is clear that the system has three linear independent solutions iff

$$a_{n-k} = \frac{1}{k}\Big( R_n^{k+1}\lambda^k - \sum_{s=0}^{k-3} R_{n-2-s}^{k-1-s} a_{n-2-s}\lambda^{k-2-s} \Big)\,,\ k = 3, 4, \ldots, n$$

$$\lambda^2 = \frac{2}{R_n^3} a_{n-2}$$

and these identities are fulfilled for $\pm\lambda$. This completes the proof. □

4.3.2. *Classification of constant sections with* 1*-dimensional algebras of symmetries.*

**Proposition 4.6.** *Let $S$ be a constant section of $\pi$ with* 1*-dimensional algebra of symmetries and let $f$ be a diffeomorphism of $\mathbb{R}^1$. Then $f(S)$ is a constant section iff $f(x) = \lambda x + \nu$, $\lambda, \nu \in \mathbb{R}$, $\lambda \neq 0$.*

*Proof.* The section $S$ has the symmetry $\partial/\partial x$. It follows that $f_*$ takes this symmetry to the symmetry $f'(\, f^{-1}(X)\,)\partial/\partial X$ of the section $f(S)$. It now is obvious that $f(S)$ is a constant iff $f' = \lambda \in \mathbb{R}$, $\lambda \neq 0$. The last is equivalent to $f(x) = \lambda x + \nu$, $\nu \in \mathbb{R}$. □

It now is clear that the problem of classification of sections of $\pi$ with 1-dimensional algebras of symmetries in a neighborhood of regular point is reduced to the classification of constant sections with 1-dimensional algebras of symmetries w.r.t. the group of all linear transformations $x \mapsto \lambda x$, $\lambda \neq 0$, of $\mathbb{R}^1$.

**Proposition 4.7.** *Let $S_1 : x \to (\, x, a_{n-2}, a_{n-3}, \dots, a_0\,)$ and $S_2 : X \to (\, X,$
$A_{n-2}, A_{n-3}, \dots, A_0\,)$ be constant sections of $\pi$ with 1-dimensional algebras of symmetries. Then $S_1$ and $S_2$ are equivalent iff there exist $\lambda \in \mathbb{R}\,,\ \lambda \neq 0$, such that*

$$a_{n-2} = \lambda^2 A_{n-2}, \ \ a_{n-3} = \lambda^3 A_{n-3}, \ \ \dots, \ \ a_0 = \lambda^n A_0. \qquad (4.2)$$

*Proof.* Suppose $S_1$ and $S_2$ are equivalent. This means that there exist a linear transformation $f(x) = \lambda x$ that takes $S_1$ to $S_2$. The point transformation $(\, f, \hat{f}\,)$ of form (1.3) generated by $f$ is defined by $X = \lambda x\,,\ Y = |\lambda|^{(n-1)/2} y$. From (2.2), we have that the lifting $(\, f, \hat{f}\,)^{(n)}$ is defined by

$$X = \lambda x\,,\ Y = |\lambda|^{(n-1)/2} y\,,\ P_1 = |\lambda|^{(n-1)/2}\lambda^{-1} p_1\,, \dots, P_n = |\lambda|^{(n-1)/2}\lambda^{-n} p_n\,.$$

It follows that $p_n = \lambda^2 A_{n-2} p_{n-2} + \lambda^3 A_{n-3} p_{n-3} + \ldots + \lambda^n A_0 y$ is the equation $\mathcal{E}_{S_1}$ obtained from $\mathcal{E}_{S_2}$ by point transformation $(\, f, \hat{f}\,)$. The statement of the proposition now is obvious. □

Let $\mathcal{I} \subset \{\, 2, 3, \dots, n \,\}\,,\ n \geq 3$, be a nonempty subset. If $\mathcal{I}$ contains odd numbers, then by $m$ we denote the minimal odd number of $\mathcal{I}$. If $\mathcal{I}$ consists of even numbers, then by $m$ we denote the minimal number of $\mathcal{I}$.

By $S^{\mathcal{I}}$ we denote a constant section with 1-dimensional algebra of symmetries $S : x \to (\, x, a_{n-2}, a_{n-3}, \dots, a_0\,)$ that satisfies $a_{n-i} \neq 0$ if $i \in \mathcal{I}$ and $a_{n-i} = 0$ otherwise. Put by definition

$$I_k(S^{\mathcal{I}}) = \begin{cases} a_{n-k}/a_{n-m}^{k/m}\,, & k \in \mathcal{I}\,,\ \text{whenever } \mathcal{I} \text{ contains odd numbers} \\ a_{n-k}/|a_{n-m}|^{k/m}\,, & k \in \mathcal{I}\,,\ \text{whenever } \mathcal{I} \text{ consists of even numbers}. \end{cases}$$

**Theorem 4.8.**

(1) *If $\mathcal{I}_1 \neq \mathcal{I}_2$, then any sections $S^{\mathcal{I}_1}\,,\ S^{\mathcal{I}_2}$ are not equivalent.*
(2) *Sections $S_1^{\mathcal{I}}\,,\ S_2^{\mathcal{I}}$ are equivalent iff*

$$\forall k \in \mathcal{I} \quad I_k(S_1^{\mathcal{I}}) = I_k(S_2^{\mathcal{I}}). \qquad (4.3)$$

*Proof.* 1) The first statement follows from relations (4.2).

2) Suppose the sections $S_1^{\mathcal{I}}\,,\ S_2^{\mathcal{I}}$ are equivalent. Then, it follows from relations (4.2) that $\lambda = a_{n-m}^{1/m}/A_{n-m}^{1/m}$ whenever $\mathcal{I}$ contains odd numbers and $\lambda = |a_{n-m}|^{1/m}/|A_{n-m}|^{1/m}$ whenever $\mathcal{I}$ consists of even numbers. Substituting this expression for $\lambda$ ( respectively for $\lambda^2$ ) in (4.2), we obtain equalities (4.3).

Conversely, suppose equalities (4.3) hold. Suppose $\mathcal{I}$ contains odd numbers. Then we define $\lambda$ by formula $\lambda = a_{n-m}^{1/m}/A_{n-m}^{1/m}$. It follows that $(a_{n-m})^{1/m} = \lambda (A_{n-m})^{1/m}$. Substituting this expression for $(a_{n-m})^{1/m}$ in (4.3), we obtain relations (4.2). If $\mathcal{I}$ consists of even numbers, then setting $\lambda = |a_{n-m}|^{1/m}/|A_{n-m}|^{1/m}$, we obtain (4.2) by analogue arguments. □

**Corollary 4.9.** *The collection of numbers* $I_k(S^{\mathcal{I}})$, $k \in \mathcal{I}$, *is a complete collection of invariants to solve the equivalence problem for sections of the form* $S^{\mathcal{I}}$.

Below, by $\mathcal{I}_1$ we denote nonempty subsets of $\{ 2, 3, \ldots, n \}$, $n \geq 3$, consisting odd numbers; by $\mathcal{I}_2$ we denote nonempty subsets of $\{ 2, 3, \ldots, n \}$, $n \geq 3$, consisting of even numbers.

By $S^{\mathcal{I}_1}_{+1}$ we denote a section of the form $S^{\mathcal{I}_1}$ with $a_{n-m} = 1$, by $S^{\mathcal{I}_2}_{+1}$ we denote a section of the form $S^{\mathcal{I}_2}$ with $a_{n-m} = 1$, and at last by $S^{\mathcal{I}_2}_{-1}$ we denote a section of the form $S^{\mathcal{I}_2}$ with $a_{n-m} = -1$.

By $\mathcal{F}$ we denote the family of all sections of the forms: $S^{\mathcal{I}_1}_{+1}$, $S^{\mathcal{I}_2}_{+1}$, and $S^{\mathcal{I}_2}_{-1}$ for all $\mathcal{I}_1$, $\mathcal{I}_2$.

From stated above, we have the following two theorems.

**Theorem 4.10.** *(Classification constant sections of* $\pi$ *with* 1*-dimensional algebras of symmetries up to equivalence.)*

1. *Any two sections from* $\mathcal{F}$ *are not equivalent.*
2. *Any constant section of* $\pi$ *with* 1*-dimensional algebra of symmetries is equivalent to some section from* $\mathcal{F}$.

**Theorem 4.11.** *(Classification of linear ODEs order* $n \geq 3$ *with* $n+2$*-dimensional algebras of point symmetries in a neighborhood of a regular point up to equivalence.)*

1. *Let* $S_1, S_2 \in \mathcal{F}$ *and* $S_1 \neq S_2$. *Then any their germs* $\{S_1\}_{p_1}$, $\{S_2\}_{p_2}$ *are not equivalent.*
2. *Let* $S$ *be a section of* $\pi$ *with* 1*-dimensional algebra of symmetries and let* $p$ *be a regular point of its domain. Then* $\{S\}_p$ *is equivalent to a germ of some section from* $\mathcal{F}$.

Thus $\mathcal{F}$ is a complete family of nonequivalent sections with 1-dimensional algebras of symmetries.

As examples, consider $\mathcal{F}$ for $n = 3, 4$.

Let $n = 3$. Then $\mathcal{I}_1 = \{1, 0\}, \{0\}$ and $\mathcal{I}_2 = \{1\}$. From (4.5), we have $\dim \operatorname{Sym} S^{\mathcal{I}_2} = 3$. Thus the family $\mathcal{F}$ consists of the following sections:

$$p_3 = a_1 p_1 + y\,,\ a_1 \in \mathbb{R} \setminus \{0\}\,,$$
$$p_3 = y\,.$$

Let $n = 4$. Then $\mathcal{I}_1 = \{2, 1, 0\}, \{2, 1\}, \{1, 0\}, \{1\}$ and $\mathcal{I}_2 = \{2, 0\}, \{0\}$. From (4.5), we have that the sections $S : x \mapsto (\ x,\ \pm p_2,\ 0,\ (1/20)\, y\ )$ of the form $S^{\mathcal{I}_2}$ have 3-dimensional algebras of symmetries. All others sections of this form have 1-dimensional algebras of point symmetries.

Thus the family $\mathcal{F}$ consists of the following sections:

$$\begin{aligned}
&p_4 = a_2 p_2 + p_1 + a_0 y\,,\ a_2, a_0 \in \mathbb{R}\setminus\{0\}\,,\\
&p_4 = b_2 p_2 + p_1\,,\ b_2 \in \mathbb{R}\setminus\{0\}\,,\\
&p_4 = \pm p_2 + c_0 y\,,\ c_0 \in \mathbb{R}\setminus\{0\}\,,\ c_0 \neq 1/20\,,\\
&p_4 = p_1 + d_0 y\,,\ d_0 \in \mathbb{R}\setminus\{0\}\,,\\
&p_4 = p_1\,,\\
&p_4 = \pm y\,.
\end{aligned}$$

4.4. **Differential invariants of linear ODEs of form** (1.2)**.** In this subsection, we calculate the algebra of scalar differential invariants of linear ODEs of form (1.2) and we solve the euivalence problem for these equations.

4.4.1. *Jet bundles.* The standard coordinates $x, a_{n-i},\ i = 3, 4, \ldots, n\,,$ on the bundle $\tau$ define in the obvious way the standard coordinates $x, a^{(r)}_{n-j},\ j = 3, 4, \ldots, n\,,\ r = 0, 1, 2, \ldots, k\,,$ on the jet bundle $\tau_k : J^k\tau \to \mathbb{R}\,,\ k = 0, 1, 2, \ldots, \infty$.

Any diffeomorphism $f^{(0)}\,,\ f \in G$, can be lifted to the Lie transformations $f^{(k)}$ of the jet bundles $J^k\tau\,,\ k = 1, 2, \ldots, \infty$, by the formula

$$f^{(k)}(\,[S]^k_p\,) = [\,f^{(0)} \circ S \circ f^{-1}\,]^k_{f(p)}\,. \tag{4.4}$$

Obviously, for any $l > m$, the diagram

$$\begin{array}{ccc}
J^l\tau & \xrightarrow{f^{(l)}} & J^l\tau\\
{\scriptstyle \tau_{l,m}}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{l,m}}\\
J^m\tau & \xrightarrow[f^{(m)}]{} & J^m\tau
\end{array}$$

is commutative (in the domains of $f^{(l)}$).

Let

$$G^{(k)} = \{\ f^{(k)} \mid f \in G\ \}\,,\quad k = 0, 1, 2, \ldots, \infty\,.$$

Let

$$G_+ = \{\ f \in G \mid f' > 0\ \}\,,\quad G_- = \{\ f \in G \mid f' < 0\ \}\,.$$

Obviously, $G_+$ is the connected component of the unit of $G$.

By $\mu$ we denote the element of $G_-$ defined by

$$\mu(x) = -x\ \forall x \in \mathbb{R}^1\,.$$

Obviously, we have

$$G = G_+ \cup G_-\,,\quad G_- = \mu \circ G_+\,,$$

The lifting $\mu^{(k)}$ is defined by

$$\mu^{(k)}(\,(\,x, a^{(r)}_{n-j}\,)\,) = (\,-x, (-1)^{j+r} a^{(r)}_{n-j}\,)\,, \tag{4.5}$$
$$j = 3, 4, \ldots, n\,,\ r = 0, 1, \ldots, k.$$

Let

$$G_+^{(k)} = \{\, f^{(k)} \mid f \in G_+ \,\}\,, \quad G_-^{(k)} = \{\, f^{(k)} \mid f \in G_- \,\}\,.$$

Obviously, $G_+^{(k)}$ is the connected component of the unit of $G^{(k)}$ and

$$G^{(k)} = G_+^{(k)} \cup G_-^{(k)}\,, \quad G_-^{(k)} = \mu^{(k)} \circ G_+^{(k)}\,.$$

The lifting of projective transformations of the base $\mathbb{R}^1$ to diffeomorphisms of $J^k\tau$ generates the lifting of any vector field $\xi \in \mathfrak{g}$ to the vector field $\xi^{(k)}$ on $J^k\tau$. By definition, $\xi^{(k)}$ is the vector field defined by the flow $f_t^{(k)}$, where $f_t$ is the flow of $\xi$. Obviously

$$(\,\tau_{l,m}\,)_*(\,\xi^{(l)}\,) = \xi^{(m)}\,, \quad l > m\,.$$

Let $\xi = \varphi(x)\partial/\partial x$ be an arbitrary element of $\mathfrak{g}$. The vector field $\xi^{(\infty)}$ is defined by the formula (see [5])[1]

$$\xi^{(\infty)} = \varphi D_x + Э_\psi\,, \tag{4.6}$$

where $D_x = \partial/\partial x + \sum_{k=0}^{\infty}\sum_{j=3}^{n} a_{n-j}^{(k+1)}\partial/\partial a_{n-j}^{(k)}$ is the operator of total derivation w.r.t. $x$; $Э_\psi = \sum_{k=0}^{\infty}\sum_{j=3}^{n} D_x^k(\,\psi_{n-j}\,)\partial/\partial a_{n-j}^{(k)}$ is the operator of evolution differentiation with generating function $\psi = (\,\psi_{n-3},\ldots,\psi_0\,)^t$. This function is defined in the following way. Let $x_1 = [S]_x^1 \in J^1\tau\,,\ x = \tau_1(x_1)$; then

$$\psi(x_1) = \begin{pmatrix} \psi_{n-3}(x_1) \\ \cdots \\ \psi_0(x_1) \end{pmatrix} = \frac{d}{dt}(\,f_t^{(0)} \circ S \circ f_t^{-1}\,)(x)\Big|_{t=0} \tag{4.7}$$

Let $S(x) = (\,x, a_{n-3}(x),\ldots,a_0(x)\,)$. Then, taking into account that $df_t/dt|_{t=0} = \varphi$ and $\varphi''' = 0$, we can calculate that

$$\psi = \begin{pmatrix} -3\,a_{n-3}\varphi' - a_{n-3}^{(1)}\varphi \\ -\frac{3(n-3)}{2}\,a_{n-3}\varphi'' - 4\,a_{n-4}\varphi' - a_{n-4}^{(1)}\varphi \\ \cdots \\ -\frac{(k-1)(n-(k-1))}{2}\,a_{n-k+1}\varphi'' - k\,a_{n-k}\varphi' - a_{n-k}^{(1)}\varphi \\ \cdots \\ -\frac{n-1}{2}\cdot 1\cdot a_1\varphi'' - n\,a_0\varphi' - a_0^{(1)}\varphi \end{pmatrix} \tag{4.8}$$

[1] Here we use the Cyrillic letter Э, which is pronounced like "e" in "ten".

It now follows from (4.6) and (4.8) that for any $k = 0, 1, 2, \ldots, \infty$,

$$\xi_0^{(k)} = \frac{\partial}{\partial x} \,. \tag{4.9}$$

$$\xi_1^{(k)} = x\frac{\partial}{\partial x} - \sum_{r=0}^{k}\sum_{j=3}^{n}(j+r)a_{n-j}^{(r)}\frac{\partial}{\partial a_{n-j}^{(r)}} \,. \tag{4.10}$$

$$\begin{aligned}\xi_2^{(k)} = x^2\frac{\partial}{\partial x} - \sum_{r=0}^{k}\sum_{j=3}^{n} \Big[\; & 2x(j+r)a_{n-j}^{(r)}\frac{\partial}{\partial a_{n-j}^{(r)}} \\ & + (j-1)(n-(j-1))a_{n-(j-1)}^{(r)}\frac{\partial}{\partial a_{n-j}^{(r)}} \\ & + (2j+r-1)ra_{n-j}^{(r-1)}\frac{\partial}{\partial a_{n-j}^{(r)}} \;\Big] \,,\end{aligned} \tag{4.11}$$

where $a_{n-2}^{(r)} = 0$.

4.4.2. *Invariant subbundles.* Let $E^i$, $i = n-3, n-4, \ldots, 0, -1$, be the subspaces of the total space $E$ of $\tau$ defined by

$$E^i = \{\, (\, x,\, a_{n-3}, a_{n-4}, \ldots, a_0\,) \in E \mid a_j = 0 \text{ if } j > i \,\} \,.$$

Consider the subbundle $\tau|_{E^i} : E^i \to \mathbb{R}$ of the bundle $\tau$.

**Proposition 4.12.** *Every subbundle $E^i$ is invariant w.r.t. $G^{(0)}$.*

*Proof.* From (4.9) – (4.11), we have that the restrictions of the vector fields $\xi_0^{(0)}, \xi_1^{(0)}, \xi_2^{(0)}$ on $E^i$ are defined by

$$\xi_0^{(0)}|_{E^i} = \frac{\partial}{\partial x} \,, \tag{4.12}$$

$$\xi_1^{(0)}|_{E^i} = x\frac{\partial}{\partial x} - (\,(n-i)a_i\frac{\partial}{\partial a_i} + \cdots + na_0\frac{\partial}{\partial a_0}\,) \,, \tag{4.13}$$

$$\begin{aligned}\xi_2^{(0)}|_{E^i} = x^2\frac{\partial}{\partial x} & - 2x(\,(n-i)a_i\frac{\partial}{\partial a_i} + \cdots + na_0\frac{\partial}{\partial a_0}\,) \\ & - (\, i(n-i)a_i\frac{\partial}{\partial a_{i-1}} + \cdots + (n-1)a_1\frac{\partial}{\partial a_0}\,) \,.\end{aligned} \tag{4.14}$$

It now is clear that $\xi_0^{(0)}|_{E^i}, \xi_1^{(0)}|_{E^i}, \xi_2^{(0)}|_{E^i}$ are vector fields tangent to $E^i$. Therefore every subbundle $E^i$ is invariant w.r.t. $G_+^{(0)}$. From (4.5), we have $\mu^{(0)}(\,E^i\,) = E^i$. This concludes the proof. □

Thus, we have the following sequence of the subbundles invariant w.r.t. $G^{(0)}$.

$$E = E^{n-3} \supset E^{n-4} \supset \ldots \supset E^0 \supset E^{-1} \,.$$

Let $E_i$, $i = n-3, n-4, \ldots, 0, -1$, be the subsets of the total space $E$ of $\tau$ defined by

$$E_i = E^i \setminus E^{i-1} \;\text{ if }\; i \geq 0 \;\text{ and }\; E_{-1} = E^{-1} \,.$$

Consider the subbundle

$$\tau^i = \tau|_{E_i} : E_i \to \mathbb{R}$$

of the bundle $\tau$.

**Corollary 4.13.** *Every subbundle $E_i$ is invariant w.r.t. $G^{(0)}$.*

Thus, $E$ is the union

$$E = E_{n-3} \cup E_{n-4} \cup \ldots \cup E_0 \cup E_{-1} \tag{4.15}$$

of nonintersecting subbundles invariant w.r.t. $G^{(0)}$.

The following proposition is needed for the sequel.

**Proposition 4.14.** *The symmetric differential $(n-i)$-form $\omega_i = a_i dx^{n-i}|_{E_i}$ on $E_i$ is invariant w.r.t. $G^{(0)}$.*

*Proof.* Let us calculate the Lie derivatives of $\omega_i$ w.r.t. vector fields $\xi_0^{(0)}|_{E_i}$,
$\xi_1^{(0)}|_{E_i}, \xi_2^{(0)}|_{E_i}$. From (4.12) – (4.14), we have

$$\xi_0^{(0)}|_{E_i}(\,\omega_i\,) = 0\,,$$

$$\xi_1^{(0)}|_{E_i}(\,\omega_i\,) = a_i\,(n-i)\,dx^{n-i} - (n-i)\,a_i dx^{n-i} = 0\,,$$

$$\xi_2^{(0)}|_{E_i}(\,\omega_i\,) = a_i\,(n-i)\,2x\,dx^{n-i} - 2x\,(n-i)\,a_i dx^{n-i} = 0\,.$$

Hence $\omega_i$ is invariant w.r.t. $G_+^{(0)}$.

It follows from (4.5) that $(\mu^{(0)})^*(\omega_i) = \omega_i$. Thus $\omega_i$ is invariant w.r.t. $G^{(0)}$. □

**Corollary 4.15.** *(The transformation law of the first nonzero component.)*

*Let $\theta_0 = (\,x, 0, \ldots, 0, a_i, \ldots, a_0\,) \in E_i$, let $f \in G$, and let $f^{(0)}(\,\theta_0\,) = (\,f(x), 0, \ldots, 0, A_i, \ldots, A_0\,) \in E_i$. Then*

$$a_i = (\,f'(x)\,)^{n-i} A_i\,.$$

4.4.3. *Scalar differential invariants of linear ODEs.* In this subsection, we calculate scalar differential invariants of linear ODEs. For a general theory of scalar differential invariants refer to [1], [11].

It was proved in subsection 4.4.2 that the bundle $\tau^i = \tau|_{E_i} : E_i \to \mathbb{R}$ is invariant w.r.t. $G^{(0)}$. It follows that the jet bundle $J^k\tau^i$ are invariant w.r.t. $G^{(k)}$, $k = 1, 2\ldots, \infty$. Hence $J^k\tau^i$ are invariant w.r.t. the subgroup $G_+^{(k)} \subset G^{(k)}$, $k = 0, 1, 2\ldots, \infty$.

A function $I \in C^\infty(J^k\tau^i)$ is called a *scalar differential invariant* of $G$ $(\,G_+\,)$ if

$$(f^{(k)})^* I = I \quad \forall f \in G\ (\,G_+\,)\,.$$

Let $I$ be a scalar differential invariant of $G$ $(\,G_+\,)$ and let $S$ be a section of $\tau^i$. By definition, put

$$I(S) = (\,j_k S\,)^* I \tag{4.16}$$

For any $f \in G$, we have

$$I(\,f(S)\,) \circ f = I(S)\,. \tag{4.17}$$

Indeed,

$I(\,f(S)\,) = (\,j_k f(S)\,)^* I = (\,j_k(f^{(0)} \circ S \circ f^{-1})\,)^* I = (\,f^{(k)} \circ j_k S \circ f^{-1}\,)^* I =$
$(\,f^{-1}\,)^* \circ (\,j_k S\,)^* \circ (\,f^{(k)}\,)^* I = (\,f^{-1}\,)^* \circ (\,j_k S\,)^* I = (\,f^{-1}\,)^* I(S) =$
$I(S) \circ f^{-1}\,.$

Let $S$ be a section of $\tau$ admitting a 1-dimensional algebra of projective symmetries. Then $I(S)$ is a constant for any scalar differential invariant $I$. Indeed, let $\xi$ be a projective symmetry of $S$ and let $f_t$ be its flow. Then

$$I(S) = I(\,f_t(S)\,) = I(\,f_t(S)\,) \circ f_t = I(S) \circ f_t\,.$$

It is not hard to prove that $I \in C^\infty(J^k\tau^i)$ is a scalar differential invariant of $G_+$ iff $I$ is a solution of the system of linear PDEs

$$\begin{cases} \bar{\xi}_0^{(k)}(I) = 0 \\ \bar{\xi}_1^{(k)}(I) = 0 \\ \bar{\xi}_2^{(k)}(I) = 0\,, \end{cases} \tag{4.18}$$

where $\bar{\xi}_0^{(k)}$, $\bar{\xi}_1^{(k)}$, $\bar{\xi}_2^{(k)}$ are the restrictions of $\xi_0^{(k)}$, $\xi_1^{(k)}$, $\xi_2^{(k)}$ to $J^k\tau^i$. From (4.9) – (4.11), we have

$$\bar{\xi}_0^{(k)} = \xi_0^{(k)}\Big|_{J^k\tau^i} = \frac{\partial}{\partial x}\,, \tag{4.19}$$

$$\bar{\xi}_1^{(k)} = \xi_1^{(k)}\Big|_{J^k\tau^i} = x\frac{\partial}{\partial x} - \sum_{r=0}^{k}\sum_{j=i}^{0}(n-j+r)a_j^{(r)}\frac{\partial}{\partial a_j^{(r)}}\,, \tag{4.20}$$

$$\begin{aligned} \bar{\xi}_2^{(k)} = \xi_2^{(k)}\Big|_{J^k\pi^i} &= x^2\frac{\partial}{\partial x} - \sum_{r=0}^{k}\sum_{j=0}^{i}\Big[\,2x(n-j+r)a_j^{(r)}\frac{\partial}{\partial a_j^{(r)}} \\ &+ (n-j-1)(j+1)a_{j+1}^{(r)}\frac{\partial}{\partial a_j^{(r)}} \\ &+ (\,2(n-j)+r-1\,)ra_j^{(r-1)}\frac{\partial}{\partial a_j^{(r)}}\,\Big]\,. \end{aligned} \tag{4.21}$$

By $\mathcal{A}_i^k$ we denote the algebra of scalar differential invariants of $G_+$ on $J^k\tau^i$. We identify $A_i^k$ with its image $(\tau_{l,k}^i)^*(\mathcal{A}_i^k)$, $l > k$. As a result we have the following filtration

$$A_i = A_i^\infty \supset \ldots \supset A_i^k \supset \ldots \supset A_i^1 \supset A_i^0.$$

By $\mathcal{D}_i^k$ we denote the distribution on $J^k\tau^i$ generated by vector fields $\bar{\xi}_0^{(k)}$, $\bar{\xi}_1^{(k)}$, $\bar{\xi}_2^{(k)}$. From (4.19) – (4.21), we have that $\dim\mathcal{D}_i^k = 2$ if $i = 0$ and $k = 0$, otherwise $\dim\mathcal{D}_i^k = 3$.

By $N_i^k$ we denote the number of functionally independent scalar differential invariant in $A_i^k$. Clearly,

$$N_i^k = \dim J^k\pi^i - \dim \mathcal{D}_i^k$$

It is easy to prove that

$$N_0^0 = 0\,, \quad N_0^1 = 0\,, \quad N_0^k = k-1 \text{ if } k \geq 2\,, \tag{4.22}$$

$$N_1^0 = 0\,, \quad N_1^k = 2k \text{ if } k \geq 1\,, \tag{4.23}$$

$$N_i^0 = i-1\,, \quad N_i^k = (k+1)(i-1) + 2k \text{ if } k \geq 1\,. \tag{4.24}$$

Consider the vector field on $J^\infty\tau^i$

$$\zeta_i = |a_i|^{-1/(n-i)}\bar{D}_x\,, \tag{4.25}$$

where $\bar{D}_x = D_x|_{J^\infty\tau^i} = \partial/\partial x + \sum\limits_{r=0}^{\infty}\sum\limits_{j=i}^{0} a_j^{(r+1)}\partial/\partial a_j^{(r)}$ is the operator of total derivation w.r.t. $x$ restricted on $J^\infty\tau^i$.

**Proposition 4.16.** *The vector field $\zeta_i$ is invariant w.r.t. $G_+^{(\infty)}$.*

*Proof.* By $\bar{\xi}_r^{(\infty)}$, $r = 0, 1, 2$, we denote the restriction of $\xi_r^{(\infty)}$ to $J^\infty\tau^i$. Let us check that $[\,\zeta_i,\ \bar{\xi}_r^{(\infty)}\,] = 0$ for all $r$.

Taking into account (4.9), we have

$$[\,\zeta_i,\ \bar{\xi}_0^{(\infty)}\,] = \left[\,|a_i|^{-1/(n-i)}\bar{D}_x,\ \frac{\partial}{\partial x}\,\right] = 0\,.$$

Taking into account (4.8), we consider the vector fields $\bar{\xi}_1^{(\infty)}$ and $\bar{\xi}_2^{(\infty)}$ in the form (4.6):

$$\begin{aligned} \bar{\xi}_1^{(\infty)} &= x\bar{D}_x + \bar{\ni}_{(\,(n-i)a_i + xa_i^{(1)}\,)}\,, \\ \bar{\xi}_2^{(\infty)} &= x^2\bar{D}_x + \bar{\ni}_{(\,(i+1)(n-i+1)a_{i+1} + 2(n-i)xa_i + x^2a_i^{(1)}\,)}\,, \end{aligned}$$

where $\bar{\ni}_\psi$ is the restriction of $\ni_\psi$ on $J^\infty\tau^i$. Now taking into account that $[\,\bar{D}_x\,,\ \bar{\ni}_\psi\,] = 0$ for any $\psi$, we easily obtain that $[\,\zeta_i,\ \bar{\xi}_1^{(\infty)}\,] = 0$ and $[\,\zeta_i,\ \bar{\xi}_2^{(\infty)}\,] = 0\,.$ □

Obviously, for any $I \in \mathcal{A}_i$, its Lie derivative $\zeta_i(I) \in \mathcal{A}_i$. Thus, $\zeta_i$ and $I$ generate the sequence $I\,,\ \zeta_i(I)\,,\ \ldots,\ \zeta_i^k(I)\,,\ \ldots$ of scalar differential invariants from $\mathcal{A}_i$.

**Theorem 4.17.** *The algebra $\mathcal{A}_i$ is generated by the following free generators*

$$\zeta_i^k(I_{i-m}),\ m = 0, 1, \ldots, i\,,\ k = 0, 1, 2, \ldots\,,$$

*where*

$$I_i = \left[ 2a_i\, a_i^{(2)} - \frac{2(n-i)+1}{n-i}\, (a_i^{(1)})^2 \right] \cdot (a_i)^{-2(n-i+1)/(n-i)}\,; \quad (4.26)$$

$$I_{i-1} = \left[ a_{i-1} - \frac{i}{2}\, a_i^{(1)} \right] \cdot |a_i|^{-(n-i+1)/(n-i)}\,; \quad (4.27)$$

*for* $2 \le m \le i$,

$$I_{i-m} = \Big[ a_{i-m} + \frac{(-1)^m}{m!} \prod_{r=n-i+1}^{n-i+m-1} \frac{(n-r)r}{(n-i)i}\, (a_i)^{1-m}\, (a_{i-1})^m + $$
$$+ \sum_{l=n-i+1}^{n-i+m-1} \frac{(-1)^{n-i+m-l}}{(n-i+m-l)!} \prod_{r=l}^{n-i+m-1} \frac{(n-r)r}{(n-i)i}\, (a_i)^{i-n+l-m} \cdot$$
$$(a_{i-1})^{n-i+m-l}\, a_{n-l} \Big] \cdot |a_i|^{-(n-i+m)/(n-i)}\,. \quad (4.28)$$

*Proof.* It is not hard to check that $I_i, \ldots, I_0$ are solutions of system (4.18).

Let $i = 0$. We have that $I_0 \in \mathcal{A}_0^2$. For any $k = 0, 1, 2, \ldots$, the invariants $I_0, \zeta_0(I_0), \zeta_0^2(I_0), \ldots, \zeta_0^k(I_0)$ belong to $\mathcal{A}_0^{k+2}$ and they are functionally independent. The number of them is equal to $(k+2)-1$. Now from (4.22), we obtain that $(k+2)-1 = N_0^{k+2}$. This concludes the proof for $i = 0$.

Suppose $i \ge 1$. We have

$$\zeta_i(I_{i-1}) = [\, -\frac{i}{2}\, |a_i|\, a_i^{(2)} + \ldots\, ] \cdot (a_i)^{-2(n-i+1)/(n-i)}$$

The manifold $J^\infty\tau^i$ has two connected components defined by the inequalities $a_i > 0$ and $a_i < 0$ respectively. Comparing $\zeta_i(I_{i-1})$ with $I_i$, we can define the scalar differential invariant $J \in \mathcal{A}_i^1$ by the formula

$$J = \begin{cases} I_i + \dfrac{4}{i}\, \zeta_i(I_{i-1}) & \text{if } a_i > 0 \\ I_i - \dfrac{4}{i}\, \zeta_i(I_{i-1}) & \text{if } a_i < 0\,. \end{cases}$$

It is easy to calculate that

$$J = \left[ \frac{4}{i}\, a_i\, a_{i-1}^{(1)} - \frac{4(n-i+1)}{i(n-i)}\, a_i^{(1)}\, a_{i-1} + \frac{1}{n-i}\, (a_i^{(1)})^2 \right] \cdot (a_i)^{-2(n-i+1)/(n-i)}\,.$$

Let $i = 1$. Then $I_{i-1}$, $J \in \mathcal{A}_i^1$ and they are functionally independent. The invariants

$$I_{i-1},\ J,\ \zeta_i(I_{i-1}),\ \zeta_i(J),\ \ldots\ \zeta_i^k(I_{i-1}),\ \zeta_i^k(J)$$

belong to $\mathcal{A}_i^{k+1}$, $k = 0, 1, 2, \ldots$, they are functionally independent, and the number of them is equal to $2(k+1)$. Now from (4.23), we obtain $2(k+1) = N_1^{k+1}$. This concludes the proof for $i = 1$.

Let $i > 1$. Then the invariants $I_{i-2}, \ldots, I_0$ are functionally independent and they belong to $\mathcal{A}_i^0$. The invariants $I_{i-2}, \ldots, I_0,\ I_{i-1},\ J$ are functionally independent and they belong to $\mathcal{A}_i^1$. At last, the invariants

$$I_{i-2},\ \ldots,\ I_0, I_{i-1},\ J,\ \ldots,\ \zeta_i^k(I_{i-2}),\ \ldots,\ \zeta_i^k(I_0),\ \zeta_i^k(I_{i-1}),\ \zeta_i^k(J)$$

are functionally independent, they belong to $\mathcal{A}_i^k$, $k = 1, 2, \ldots$, and the number of them is equal to $(k+1)(i-2)+2k$. Now from (4.24), we obtain that $(k+1)(i-1)+2k = N_i^k$. This concludes the proof for $i > 1$. □

From (4.5), we obtain

**Remark 4.18.** *The invariant*

$$I_i = \left[ 2a_i\, a_i^{(2)} - \frac{2(n-i)+1}{n-i}\, (a_i^{(1)})^2 \right] \cdot (a_i)^{-2(n-i+1)/(n-i)}$$

*is an invariant of the group $G$.*

4.4.4. *The equivalence problem of linear ODEs.* Let

$$S_1(x) = (\,0,\ \ldots,\ 0,\ a_i(x),\ \ldots,\ a_0(x)\,)$$

and

$$S_2(X) = (\,0,\ \ldots,\ 0,\ A_i(X),\ \ldots,\ A_0(X)\,)$$

be sections of $\tau^i$ in neighborhoods of points $p \in \mathbb{R}$ and $P \in \mathbb{R}$ respectively.

Sections $S_1$ and $S_2$ are *locally equivalent at $(\,p, P\,)$ w.r.t. $G_+$ $(\,G\,)$* if there exist $f \in G_+$ $(\,G\,)$ and neighborhoods $V$ of $p$ and $U$ of $P$ that $f(p) = P$ and $f(S_1|_V) \overset{def}{=} f^{(0)} \circ S_1|_V \circ f^{-1} = S_2|_U$.

**Theorem 4.19.** *Sections $S_1$ and $S_2$ of $\tau^i$ are locally equivalent at $(\,p, P\,)$ w.r.t. $G_+$ iff the following conditions hold:*

(1) $a_i(p) \cdot A_i(P) > 0\,,$

(2) *the solution $f$ of the Cauchy problem*

$$f' = |a_i(x)|^{1/(n-i)} \cdot |A_i(f(x))|^{-1/(n-i)}\,,\ f(p) = P \tag{4.29}$$

*satisfies to the equations*

$$I_m(S_2) \circ f = I_m(S_1)\,,\ \ m = i, i-1, \ldots, 0 \tag{4.30}$$

*in some neighborhood of p.*

*Proof.* Suppose $S_1$ and $S_2$ are locally equivalent at $(\,p, P\,)$ w.r.t. $G_+$. Then there exist $f \in G_+$ and neighborhoods $V$ of $p$ and $U$ of $P$ that $f(p) = P$ and $f(S_1|_V) = S_2|_U$. Consider the symmetric differential $(n-i)$-form $\omega_i$ on $E_i$ (see proposition 4.14). We have

$$f^*(\,S_2^*(\omega_i)\,) = S_1^*(\omega_i)\,. \tag{4.31}$$

Indeed,

$$f^*(\,f(S_1)^*(\omega_i)\,) = f^*(\,(f^{(0)} \circ S_1 \circ f^{-1})^*(\omega_i)\,) = S_1^*(\,(f^{(0)})^*(\omega_i)\,) = S_1^*(\omega_i)\,.$$

Equality (4.31) means that

$$a_i(x) = (f')^{n-i} A_i(f(x)) \,.$$

It follows both that $a_i(p) \cdot A_i(P) > 0$ and that $f$ is a solution of Cauchy problem (4.29). Further, from (4.17), we have that equations (4.30) hold.

Conversely, let $a_i(p) \cdot A_i(P) > 0$, let $f$ be a solution of Cauchy problem (4.29), and let $f$ be a solution of equations (4.30).

Let us show that $f \in G_+$. From (4.29), we can obtain $f''$ and $f'''$ in terms of $a_i$, $A_i$ and their 1-st and 2-nd derivatives:

$$f'' = \frac{1}{r}\left[ \, |a_i|^{\frac{1-r}{r}} |A_i|^{\frac{-1}{r}} \operatorname{sgn}(a_i) a_i' - |a_i|^{\frac{2}{r}} |A_i|^{\frac{-r-2}{r}} \operatorname{sgn}(A_i) A_i' \right] , \quad (4.32)$$

$$\begin{aligned} f''' = \frac{1}{r}\Bigg[ & \frac{1-r}{r} |a_i|^{\frac{1-2r}{r}} |A_i|^{\frac{-1}{r}} (a_i')^2 + |a_i|^{\frac{1-r}{r}} |A_i|^{\frac{-1}{r}} \operatorname{sgn}(a_i) a_i'' \\ & - \frac{3}{r} |a_i|^{\frac{2-r}{r}} |A_i|^{\frac{-r-2}{r}} a_i' A_i' + \frac{2+r}{r} |a_i|^{\frac{3}{r}} |A_i|^{\frac{-2r-3}{r}} (A_i')^2 \\ & - |a_i|^{\frac{3}{r}} |A_i|^{\frac{-r-3}{r}} \operatorname{sgn}(A_i) A_i'' \Bigg] , \end{aligned} \quad (4.33)$$

where $r = n - i$. Substituting expressions (4.29),(4.32), and (4.33) for $f'$, $f''$, and $f'''$ in the left side of equation (3.4), we obtain

$$2\, f''' \, f' - 3\,(f'')^2 = \frac{1}{n-i} \, |a_i|^{4/(n-i)} |A_i|^{-2/(n-i)} ( \, I_i(S_1) - I_i(S_2) \circ f \,) = 0 \,.$$

Thus, $f \in G_+$.

Let $S_3 = f(S_1)$. Then $I_m(S_3) \circ f = I_m(\, f(S_1)\,) \circ f = I_m(S_1) = I_m(S_2) \circ f$, $m = i, i-1, \ldots, 0$. Hence

$$I_m(S_3) = I_m(S_2) \,, \; m = i, i-1, \ldots, 0 \,.$$

Let $S_3 = (\, 0, \, \ldots, \, 0, \, B_i, \, \ldots, \, B_0 \,)$. Then obviously, $B_i = (f')^{-(n-i)} a_i = A_i$ in some neighborhood of $P$. It now follows from $I_{i-1}(S_3) = I_{i-1}(S_2)$ that $B_{i-1} = A_{i-1}$ in this neighborhood. From $I_{i-2}(S_3) = I_{i-2}(S_2)$, we have $B_{i-2} = A_{i-2}$ in this neighborhood and so on. Thus $S_3 = S_2$ in some neighborhood of $P$. □

**Corollary 4.20.** *Sections $S_1, S_2$ of $\tau^i$ are locally equivalent at the $(\, p, P\,)$ w.r.t. $G$ iff $S_1$ local equivalent w.r.t. $G_+$ either to $S_2$ at $(\, p, P\,)$ or to $\mu(S_2)$ at $(\, p, -P\,)$.*

**Corollary 4.21.** *Let the invariants $I_m(S_1), I_m(S_2)$, $m = i, i-1, \ldots, 0$ be constants. Then $S_1, S_2$ are locally equivalent at $(\, p, P\,)$ w.r.t. $G_+$ iff the following conditions hold:*

(1) $a_i(p) \cdot A_i(P) > 0$,
(2) $I_m(S_1) = I_m(S_2)$, $m = i, i-1, \ldots, 0$.

**Proposition 4.22.** *Let $S$ be a section of $\tau^i$; then* $\dim \operatorname{Prj} S = 1$ *iff* $I_i(S), I_{i-1}(S), \ldots, I_0(S)$ *are constants.*

*Proof.* The necessary is proved in the beginning of this subsection. Prove the sufficiency. Let $S(x) = (\, x, 0, \ldots, 0, a_i(x), \ldots, a_0(x)\,)$ and invariants $I_i(S), I_{i-1}(S), \ldots, I_0(S)$ are constants.

From proposition 3.2, we have that a vector field $\varphi(x)\partial/\partial x \in \mathfrak{g}$ is a symmetry of section $S$ iff $\varphi(x)$ is a solution of system (2.11). In our case, this system has the following form:

$$\begin{cases} \varphi''' = 0 \\ (n-i)a_i\varphi' + a_i'\varphi = 0 \\ \dfrac{(n-j-1)(j+1)}{2}a_{j+1}\varphi'' + (n-j)a_j\varphi' + a_j'\varphi = 0\,, \\ j = i-1, i-2, \ldots, 0\,. \end{cases} \tag{4.34}$$

It follows from the second equation of this system that $\varphi = C|a_i|^{-1/(n-i)}$, $C \in \mathbb{R}$.

From the identities $d\,I_m(S)/dx \equiv 0\,,\ m = i, i-1, \ldots, 0$, we can obtain by direct calculations that $|a_i|^{-1/(n-i)}$ is a solution of system (4.34). Thus, the vector field $|a_i|^{-1/(n-i)}\partial/\partial x$ is a symmetry of the section $S$. □

4.5. **Classification of linear ODEs with $n+1$ – dimensional algebra of point symmetries.** In this subsection, we use the scalar differential invariants obtained in the previous subsection to classify linear ODEs (1.1) with $n+1$ – dimensional algebra of point symmetries in a neighborhood of a regular point up to a transformation (1.3).

4.5.1. *Regular germs.* Let $S$ be a section of $\pi$, let $p$ be an arbitrary point of the domain of $S$, and let $f$ be a Lagguere-Forsyth transformation of $S$ defined in a neighborhood of $p$. We say that $p$ is a *point of class $i$ for $S$ (for the ODE $\mathcal{E}_S$)* if there exist a neighborhood $U'$ of $f(p)$ and subbundle $E_i$ of $\tau$ (see section 4.4.2) with $\operatorname{Im} f(S)|_{U'} \subset E_i$. It is easy to prove that a point of class $i$ is well defined.

We say that $S$ is *a section of class $i$* if every point of the domain of $S$ is a point of class $i$.

Clearly, if $p$ is a point of class $i$ for $S$ then, for some neighborhood $U$ of $p$, $S|_U$ is a section of class $i$.

Let $S$ be a section of class $i$ with $\dim \operatorname{Sym} S = 0$, let $f$ be a Laguerre-Forsyth transformation of $S$, let $S' = f(S)$, and let $I_i, I_{i-1}, \ldots, I_0$ be the generators of the algebra of scalar differential invariants $\mathcal{A}_i$ defined in theorem 4.17. Consider the smooth functions $I_i(S'), I_{i-1}(S'), \ldots, I_0(S')$ in $\mathbb{R}^1$ (see formula (4.16)). From $\dim \operatorname{Sym} S = 0$ we have $\dim \operatorname{Sym} S' = 0$. It follows from proposition 4.22 that there exist integer $j\,,\ i \le j \le 0$, such that $I_j(S')$ is not constant. We say that a point $p$ of domain of $S$ is *regular point of $S$ (of the ODE $\mathcal{E}_S$)* if $dI_j(S')|_{f(p)} \neq 0$.

Obviously, the set of all regular points of a section of class $i$ is an everywhere dense subset of the domain of this section.

We say that the germ $\{S\}_p$ with $\dim \mathrm{Sym}\, S = 0$ is *a regular germ of class $i$* if $p$ is both a point of class $i$ for $S$ and a regular point for $S$.

By $\tilde{\mathfrak{F}}_i$ , $i = n-3, n-4, \dots, 0$ we denote the set of all regular germs $\{S\}_0$ of class $i$.

It is easy to prove that $\tilde{\mathfrak{F}}_i$ is invariant w.r.t. diffeomorphisms from $\Gamma_0$.

4.5.2. *Classification of regular germs.* Let $\{S\}_0 \in \tilde{\mathfrak{F}}_i$, let $f$ be a Laguerre-Forsyth transformation of $S$, let $S' = f(S)$, and let $p_0 = f(0)$. By definition put

$$m = \max\{\, j \in \{i, i-1, \dots, 0\} \mid dI_j(S')|_{p_0} \neq 0 \,\}$$

From (4.17), we have that the number $m$ is an invariant of the action of $\Gamma$.

It follows from $dI_m(S')|_{p_0} \neq 0$ that there exists a neighborhood $U$ of $p_0$ such that we can consider $I_m(S')|_U$ as an element of $\Gamma$. Let

$$\tilde{f} = I_m(S')|_U - I_m(S')(p_0)\,.$$

Obviously, $\tilde{f} \in \Gamma$. Below, it will be useful to represent the function $\tilde{f}$ in the form

$$\tilde{f} = R_{I_m(S')(p_0)} \circ I_m(S')|_U\ ,$$

where the function $R_a$ is defined by $R_a : x \mapsto x - a$.

We say that the germ $\{\, \tilde{f}(S')\,\}_0 \in \tilde{\mathfrak{F}}_i$ is the *canonical form of the germ* $\{S\}_0$ . It follows from the next theorem that the canonical form of $\{S\}_0$ is well defined.

**Theorem 4.23.** *Let $\{S_1\}_0, \{S_2\}_0 \in \tilde{\mathfrak{F}}_i$. Then $\{S_1\}_0$ and $\{S_2\}_0$ are equivalent iff their canonical forms are the same.*

*Proof.* Let $\{S_1\}_0$ and $\{S_2\}_0$ are equivalent. It follows that there exist $g \in \Gamma_0$ with $\{g(S_1)\}_0 = \{S_2\}_0$. Let $f_1$ and $f_2$ be Laguerre-Forsyth transformations in neigborhood of $0 \in \mathbb{R}^1$ for $S_1$ and $S_2$ respectively. Let $S_1' = f_1(S_1)\,,\ S_2' = f_2(S_2)$ and let $p_1 = f_1(0)\,,\ p_2 = f_2(0)$. Then in some neighborhood of $p_2$, we have

$$\begin{aligned}
\tilde{f}_2 &= I_m(S_2') - I_m(S_2')(p_2)\\
&= I_m(\,(f_2 \circ g \circ f_1^{-1})(S_1')\,) - I_m(\,(f_2 \circ g \circ f_1^{-1})(S_1')\,)(p_2)\\
&= I_m(S_1') \circ (f_1 \circ g^{-1} \circ f_2^{-1}) - I_m(S_1')(p_1)\\
&= R_{I_m(S_1')(p_1)} \circ I_m(S_1') \circ (f_1 \circ g^{-1} \circ f_2^{-1})\,.
\end{aligned}$$

Hence in the correspondence neighborhood of $0 \in \mathbb{R}^1$, we have

$$\begin{aligned}
\tilde{f}_2(S_2') &= R_{I_m(S_1')(p_1)} \circ I_m(S_1') \circ (f_1 \circ g^{-1} \circ f_2^{-1})(\,(f_2 \circ g \circ f_1^{-1})(S_1')\,)\\
&= (\, R_{I_m(S_1')(p_1)} \circ (\, I_m(S_1')\,)(S_1') = \tilde{f}_1(S_1')
\end{aligned}$$

The sufficiency is obvious. □

Let $\mathcal{F}_i$ be the set of canonical forms of all germs containing in $\tilde{\mathcal{F}}_i$ and let $\mathcal{F} = \bigcup_{i=0}^{n-3} \mathcal{F}_i$. Obviously, we get the following theorem.

**Theorem 4.24.** *(Classification of linear ODEs with $n+1$–dimensional algebras of point symmetries in neighborhoods of regular points up to equivalence.)*

(1) *Any two germs from $\mathcal{F}$ are not equivalent.*
(2) *Any regular germ of section of $\pi$ is equivalent to some germ from $\mathcal{F}$.*

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