

**On one-parametric family of Bäcklund  
autotransformations for the Liouville  
equation**

by

Arthemy V. Kiselev

Available via INTERNET:  
<http://diffiety.ac.ru/>

**The Diffiety Institute**  
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

# On one-parametric family of Bäcklund autotransformations for the Liouville equation

Arthemey V. Kiselev

ABSTRACT. Scaling symmetry of the Liouville equation is proved to provide a one-parametric family of one-dimensional non-abelian coverings determining a Bäcklund autotransformation. Deformation of the structural element is shown to be equal to the Frölicher-Nijenhuis bracket of the symmetry lifting and the structural element of the coverings. An intriguing by-formula in total derivatives is obtained.

**Mathematics Subject Classification (2000):** 35Q53, 58J72, 32G08, 13N15.

This paper illustrates new cohomological concepts [1, 2] in the theory of Bäcklund transformations between PDE. Notation and all definitions follow [1, 3]. First, let us introduce the notion of Bäcklund (auto)transformation in terms of coverings over PDE.

**1.** Let  $\mathcal{E}_i \subset J^{k_i}(\pi_i)$ ,  $i = 1, 2$ , be two differential equations and  $\tau_i: \tilde{\mathcal{E}} \rightarrow \mathcal{E}_i$  be coverings with the same total space  $\tilde{\mathcal{E}}$ . Then the diagram

$$(1) \quad \mathcal{E}_1^\infty \xleftarrow{\tau_1} \tilde{\mathcal{E}} \xrightarrow{\tau_2} \mathcal{E}_2$$

is called a *Bäcklund transformation*  $\mathcal{B}(\tilde{\mathcal{E}}, \tau_i, \mathcal{E}_i)$  between the equations  $\mathcal{E}_i$ . Diagram (1) is called a *Bäcklund autotransformation* if  $\mathcal{E}_1^\infty = \mathcal{E}_2^\infty = \mathcal{E}^\infty$ .

*Remark 1.* Let  $\tau_j: \tilde{\mathcal{E}}_j \rightarrow \mathcal{E}_j$ ,  $j = 1, 2$ , be two coverings and  $\mu: \tilde{\mathcal{E}}_1 \rightarrow \tilde{\mathcal{E}}_2$  be a diffeomorphism that maps the Cartan distribution  $\mathcal{C}_{\tau_1}D(\tilde{\mathcal{E}}_1)$  into  $\mathcal{C}_{\tau_2}D(\tilde{\mathcal{E}}_2)$ . Then the diagram  $\mathcal{B}(\tilde{\mathcal{E}}_j, \tau_1, \tau_2 \circ \mu, \mathcal{E}_j)$  is also a Bäcklund transformation between the equations  $\mathcal{E}_j$ .

*Remark 2.* Let  $\tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E}^\infty$  be a covering and  $\mu: \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$  be a nontrivial diffeomorphism of manifolds preserving Cartan distribution, e.g., a discrete symmetry that cannot be restricted to  $\mathcal{E}^\infty$ . Then the diagram

$$(2) \quad \begin{array}{ccc} \tilde{\mathcal{E}} & \xrightarrow{\mu} & \tilde{\mathcal{E}} \\ \tau \searrow & & \swarrow \tau \\ & \mathcal{E}^\infty & \end{array}$$

is also Bäcklund autotransformation for  $\mathcal{E}$ . In the sequel, this construction is used for generating Bäcklund autotransformations for the Liouville equation.

**2.** The hyperbolic Liouville equation  $\mathcal{E}$  is

$$(3) \quad \mathcal{E} = \{F \equiv u_{xy} - \exp(2u) = 0\}.$$

Consider the covering structure in  $\tau_t: \tilde{\mathcal{E}}_t \rightarrow \mathcal{E}^\infty$  provided by the extended total derivatives

$$(4) \quad \tilde{D}_x = \bar{D}_x + \tilde{u}_x \partial / \partial \tilde{u}, \quad \tilde{D}_y = \bar{D}_y + \tilde{u}_y \partial / \partial \tilde{u}, \quad [\tilde{D}_x, \tilde{D}_y] = 0,$$

if the partial derivatives of the nonlocal variable  $\tilde{u}$  w.r.t.  $x$  and  $y$  are

$$(5) \quad \tilde{u}_x = u_x + \exp(-t) \cdot \exp(\tilde{u} + u), \quad \tilde{u}_y = -u_y + 2 \exp(t) \cdot \sinh(\tilde{u} - u).$$

Let the diffeomorphism  $\mu$  be the swapping  $u \leftrightarrow \tilde{u}$  of the fiber variable  $u$  and the nonlocal variable  $\tilde{u}$  combined with  $x \mapsto -x$  and  $y \mapsto -y$ . Then diagram (2) determines Bäcklund autotransformation  $\mathcal{B}(\tilde{\mathcal{E}}_t, \tau_t, \tau_t \circ \mu, \mathcal{E})$  for Eq. (3), and for the Liouville equation, the equations  $\tilde{\mathcal{E}}_t$  of Bäcklund autotransformation [4] are

$$(6) \quad (\tilde{u} - u)_x = \exp(-t) \cdot \exp(\tilde{u} + u),$$

$$(7) \quad (\tilde{u} + u)_y = 2 \exp(t) \cdot \sinh(\tilde{u} - u).$$

**Proposition 1.** *Let  $t_1 \neq t_2$ . Then the coverings  $\tau_{t_1}$  and  $\tau_{t_2}$  are non-equivalent.*

*Proof.* Consider the Whitney sum  $\tau_{t_1} \oplus \tau_{t_2}$  of the coverings  $\tau_{t_1}$  and  $\tau_{t_2}$  with nonlocal variables  $\tilde{u}$  and  $\tilde{u}'$  respectively. These coverings are equivalent if their Whitney sum is reducible. Following [1], we verify this condition in local coordinates and prove that the system

$$(8) \quad \left( \bar{D}_x + (u_x + \exp(\tilde{u} + u - t_1)) \frac{\partial}{\partial \tilde{u}} + (u_x + \exp(\tilde{u}' + u - t_2)) \frac{\partial}{\partial \tilde{u}'} \right) \phi = 0,$$

$$(9) \quad \left( \bar{D}_y + (-u_y + 2 \exp(t_1) \sinh(\tilde{u} - u)) \frac{\partial}{\partial \tilde{u}} + (-u_y + 2 \exp(t_2) \sinh(\tilde{u}' - u)) \frac{\partial}{\partial \tilde{u}'} \right) \phi = 0$$

has only constant solution  $\phi = \text{const}$  for  $t_1 \neq t_2$ . Put  $u_k \equiv \partial^k u / \partial x^k$  and  $u_{\bar{k}} \equiv \partial^k u / \partial y^k$  for any  $k \in \mathbb{N}$ . The characteristics of Eq. (8) are

$$\frac{d\tilde{u}}{u_x + \exp(\tilde{u} + u - t_1)} = \frac{d\tilde{u}'}{u_x + \exp(\tilde{u}' + u - t_2)} = \frac{du}{u_1} = \frac{du_k}{u_{k+1}} = dq_1,$$

$$\frac{d\tilde{u}}{-u_y + 2 \exp(t_1) \sinh(\tilde{u} - u)} = \frac{d\tilde{u}'}{-u_y + 2 \exp(t_2) \sinh(\tilde{u}' - u)} = \frac{du}{u_1} = \frac{du_{\bar{k}}}{u_{\bar{k}+1}} = dq_2,$$

$k \in \mathbb{N}$ . We have

$$dq_1 = \frac{d(\tilde{u} - u)}{\exp(\tilde{u} + u - t_1)} = \frac{d(\tilde{u}' - u)}{\exp(\tilde{u}' + u - t_2)},$$

whence

$$\frac{d(\tilde{u} - \tilde{u}')}{\exp(\tilde{u} - t_1) - \exp(\tilde{u}' - t_2)} = 0.$$

Consequently,  $\tilde{u}' - \tilde{u} = \delta = \text{const}$  and  $\delta$  depends at most on  $t_1$  and  $t_2$ . Then we have  $\tilde{u}_x = \tilde{u}'_x$  and  $\tilde{u}_y = \tilde{u}'_y$  as equations for  $\delta$ . The first equation implies  $\delta = t_1 - t_2$  and the second one gives

$$\exp(\delta) \sinh(\tilde{u}' + \delta - u) = \sinh(\tilde{u}' - u).$$

Using hyperbolic-trigonometry formulas, we obtain

$$\exp(\delta) \sinh(\tilde{u}' - u) \cosh(\delta) + \exp(\delta) \cosh(\tilde{u}' - u) \sinh(\delta) = \sinh(\tilde{u}' - u).$$

Note that  $\exp(\delta) \cosh(\delta) - 1 = \exp(\delta) \sinh(\delta)$ . Finally, we emphasize that the equation  $\sinh(\tilde{u}' - u) = \cosh(\tilde{u}' - u)$  has no roots in  $\mathbb{C}$ . Therefore, system (8)-(9) has only constant solution  $\phi = \text{const}$  for  $t_1 \neq t_2$ .  $\square$

Proposition 1 means that the parameter  $t$  cannot be eliminated by a coordinate transformation if the later does not depend on  $t$  itself.

3. Consider the scaling symmetry of base coordinates

$$X^0 = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

It corresponds to the generating function  $\varphi = xu_x - yu_y$ . The scaling symmetry can be extended to the entire  $\mathcal{E}^\infty$ :

$$(10) \quad \hat{X} = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \sum_{k \geq 1} k u_k \frac{\partial}{\partial u_k} - \sum_{k \geq 1} k u_{\bar{k}} \frac{\partial}{\partial u_{\bar{k}}}.$$

*Remark 3.* The diffeomorphisms  $A_t = \exp(t\hat{X})$  form an abelian group. Really, we see that  $x(t) = \exp(-t)x(0)$ ,  $y(t) = \exp(t)y(0)$ ,  $u(t) = u(0)$ ,  $u_k(t) = \exp(kt)u_k(0)$ ,  $u_{\bar{k}}(t) = \exp(-kt)u_{\bar{k}}(0)$ ,  $k \geq 1$ . Obviously,  $A_0 = \text{id}$  and  $A_{t_1} \circ A_{t_2} = A_{t_2} \circ A_{t_1} = A_{t_1+t_2}$ .

**Proposition 2.** *The symmetry  $\hat{X}$  cannot be extended to a symmetry of the covering equation  $\tilde{\mathcal{E}}_t$ .*

*Proof.* Assume the converse. By  $\tilde{\alpha}_\varphi$  we denote the evolutionary vector field  $\sum_\sigma \tilde{D}_\sigma(\varphi) \cdot \partial/\partial u_\sigma$  on  $\tilde{\mathcal{E}}_t$ ,  $\varphi \in C^\infty(\tilde{\mathcal{E}}_t)$ , and  $\tilde{\ell}_F(\varphi) = \tilde{\alpha}_\varphi(F)$ . Suppose there is a smooth function  $a \in C^\infty(\tilde{\mathcal{E}}_t)$  such that the system

$$(11) \quad \tilde{\ell}_F(\varphi) = 0, \quad \tilde{D}_{x^i}(a) = \tilde{\alpha}_{\varphi,a}(\tilde{u}_{x^i}) \equiv (\tilde{\alpha}_\varphi + a \partial/\partial \tilde{u})(\tilde{u}_{x^i}), \quad x^1 \equiv x, \quad x^2 \equiv y,$$

holds. This means that the field  $\tilde{\alpha}_{\varphi,a}$  is a local symmetry of the covering equation  $\tilde{\mathcal{E}}_t$  and  $\hat{X}$  is constructively extended onto  $\tilde{\mathcal{E}}_t$ . Nevertheless, system (11) is not compatible since

$$\tilde{D}_x \circ \tilde{D}_y(a) \neq \tilde{D}_y \circ \tilde{D}_x(a).$$

Really,  $\tilde{D}_x \circ \tilde{D}_y(a) - \tilde{D}_y \circ \tilde{D}_x(a)$  does not depend on  $a$  at all and equals

$$\begin{aligned} & x u_x^2 \exp(t + u - \tilde{u}) + u_x y u_y \exp(t + \tilde{u} - u) - x u_x \exp(2\tilde{u}) - u_x y u_y \exp(t + u - \tilde{u}) \\ & - 2y u_y \exp(2t + \tilde{u} + u) + 2x u_x \exp(2t + \tilde{u} + u) - x u_x^2 \exp(t + \tilde{u} - u) + x \exp(2u) u_x \\ & - y \exp(2u) u_y + 2 \exp(t) x u_x^2 + y u_y \exp(2\tilde{u}) - 2 \exp(t) y u_y u_x \neq 0. \end{aligned}$$

This contradiction concludes the proof.  $\square$

Thus, the scaling symmetry  $\hat{X}$  is a  $\tau_t$ -shadow only and  $\hat{X}$  provides a family of covering equations  $\tilde{\mathcal{E}}_t$  over  $\mathcal{E}^\infty$ ,  $\tilde{\mathcal{E}}_t$  parametrized by  $t \in \mathbb{R}$ .

4. In local coordinates, the structural element  $U_t$  of the covering diffiety  $\tilde{\mathcal{E}}_t$  is

$$(12) \quad U_t = \sum_\sigma d_C(u_\sigma) \otimes \frac{\partial}{\partial u_\sigma} + (d\tilde{u} - (u_x + \exp(\tilde{u} + u - t)) dx + (u_y - 2 \exp(t) \sinh(\tilde{u} - u)) dy) \otimes \frac{\partial}{\partial \tilde{u}},$$

where  $d_C$  is the Cartan differential,  $d_C u_\sigma = du_\sigma - \sum_i u_{\sigma+1_i} dx^i$ .

By  $[\cdot, \cdot]^{\text{FN}}$  we denote the Frölicher-Nijenhuis bracket [3, 2]:  $[\Omega, \Theta]^{\text{FN}}(f) = L_\Omega(\Theta(f)) - (-1)^{\mu\nu} \cdot L_\Theta(\Omega(f))$ , where  $\Omega, \Theta \in D(\Lambda^*(\mathcal{E}))$ ,  $f \in C^\infty(\mathcal{E})$ , degrees  $\mu, \nu$  are  $\mu = \deg \Omega$ ,

$\nu = \deg \Theta$ ;  $L_\Omega = [i_\Omega, d]: \Lambda^k(\mathcal{E}) \rightarrow \Lambda^{k+\deg \Omega}(\mathcal{E})$  is the Lie derivative, and  $i_\Omega: \Lambda^k(\mathcal{E}) \rightarrow \Lambda^{k+\deg \Omega-1}(\mathcal{E})$  is the inner product.

**Theorem** ([2]). *Let  $\tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$  be a covering and  $A_t: \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$  be a smooth family of diffeomorphisms such that  $A_0 = \text{id}$  and  $\tau_t = \tau \circ A_t: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$  is a covering for any  $t \in \mathbb{R}$ . Then the structural element  $U_{\tau_t}$  evolves by*

$$(13) \quad \frac{dU_{\tau_t}}{dt} = \llbracket \hat{X}_t, U_{\tau_t} \rrbracket^{\text{FN}},$$

where  $\hat{X}_t$  is a  $\tau_t$ -shadow for any  $t \in \mathbb{R}$ .

If the covering  $\tau_t$  corresponds to Bäcklund autotransformation (6)-(7) for the Liouville equation (3), then we have

$$(14) \quad \frac{dU_t}{dt} = (\exp(\tilde{u} + u - t) dx - 2 \exp(t) \sinh(\tilde{u} - u) dy) \otimes \frac{\partial}{\partial \tilde{u}}.$$

Now we claim that the scaling symmetry  $\hat{X}$  is the  $\tau_t$ -shadow such that Eq. (13) holds, where  $U_{\tau_t}$  is the structural element  $U_t$  (12) of the covering  $\tau_t$  defined in (4) and  $U_t$  evolves according to (14). We need Lemmas 1-4 to prove it.

We say that  $i$  is the degree of  $\Omega$ , if  $\Omega \in D(\Lambda^i(\tilde{\mathcal{E}}))$ . If  $\tilde{\mathcal{E}}$  is a finite-dimensional manifold, we have an isomorphism  $D(\Lambda^*(\tilde{\mathcal{E}})) \simeq \Lambda^*(\tilde{\mathcal{E}}) \otimes D(\tilde{\mathcal{E}})$  and thus any derivation  $\Omega \in D(\Lambda^*(\tilde{\mathcal{E}}))$  is representable as a finite sum, the summands are  $\Omega = \omega \otimes X$ , where  $\omega \in \Lambda^*(\tilde{\mathcal{E}})$  and  $X \in D(\tilde{\mathcal{E}})$ . For such elements, the Frölicher-Nijenhuis bracket is

$$(15) \quad \begin{aligned} \llbracket \omega \otimes X, \theta \otimes Y \rrbracket^{\text{FN}} &= \omega \wedge \theta \otimes [X, Y] + \omega \wedge L_X(\theta) \otimes Y + (-1)^i d\omega \wedge (X \lrcorner \theta) \otimes Y - \\ &\quad - (-1)^{ij} \theta \wedge L_Y(\omega) \otimes X - (-1)^{(i+1)j} d\theta \wedge (Y \lrcorner \omega) \otimes X, \end{aligned}$$

where  $X, Y \in D(\tilde{\mathcal{E}})$ ,  $\omega \in \Lambda^i(\tilde{\mathcal{E}})$  and  $\theta \in \Lambda^j(\tilde{\mathcal{E}})$ .

**Lemma 1.**  $\llbracket \hat{X}, U_t \rrbracket^{\text{FN}} \lrcorner d\tilde{u} = (dU_t/dt) \lrcorner d\tilde{u}$ .

**Lemma 2.**  $\llbracket \hat{X}, U_t \rrbracket^{\text{FN}} \lrcorner dx = \llbracket \hat{X}, U_t \rrbracket^{\text{FN}} \lrcorner dy = \llbracket \hat{X}, U_t \rrbracket^{\text{FN}} \lrcorner du = 0$ .

*Proof.* The proof of Lemmas 1 and 2 is straightforward by multiple using of (15) and thus omitted.  $\square$

However, the calculation of coefficients of  $\llbracket \hat{X}, U_t \rrbracket^{\text{FN}}$  at  $\partial/\partial u_k$  or  $\partial/\partial u_{\bar{k}}$ ,  $k \geq 1$ , is not so transparent.

*Remark 4.* Let  $u(x)$  and  $f(u)$  be smooth functions,  $D_x$  be the total derivative w.r.t.  $x$ ; take an integer  $n > 0$  and a positive integer  $l \leq n - 1$ . Then

$$(16) \quad D_x \left( \frac{\partial}{\partial u_l} D_x^{n-1}(f(u)) \right) = \frac{\partial}{\partial u_l} D_x^n(f(u)) - \frac{\partial}{\partial u_{l-1}} D_x^{n-1}(f(u)).$$

*Corollary 1.* Moreover,

$$(17) \quad (n+1)u_{n+1} \frac{\partial}{\partial u_{n+1}} D_x^{n+1}(f(u)) = (n+1)u_{n+1} \frac{\partial}{\partial u_n} D_x^n(f(u)) = (n+1)u_{n+1} \cdot f'(u).$$

**Lemma 3.** *Let  $u(x)$  and  $f(u)$  be smooth functions,  $D_x$  be the total derivative w.r.t.  $x$ . Denote  $u_k \equiv D_x^k(u(x))$ ,  $k \geq 0$ ,  $u_0 \equiv u$ . Then for any integer  $n \geq 0$  the relation*

$$(18) \quad n \cdot D_x^n(f(u)) = \sum_{m=1}^n m u_m \frac{\partial}{\partial u_m} D_x^n(f(u))$$

holds.

*Proof.* We prove (18) by induction on  $n$ . For  $n = 0$  relation (18) is valid. For  $n \geq 0$  one has

$$(n+1) D_x^{n+1}(f(u)) = D_x(n D_x^n(f(u)) + D_x^n(f(u))) =$$

by the inductive assumption,

$$= D_x \left( \sum_{m=1}^n m u_m \frac{\partial}{\partial u_m} D_x^n(f(u)) + D_x^n(f(u)) \right) =$$

by the Leibnitz rule,

$$= \sum_{m=1}^n m u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) + \sum_{m=1}^n m u_m D_x \frac{\partial}{\partial u_m} D_x^n(f(u)) + D_x D_x^n(f(u)) =$$

by (16) applied to the second sum,

$$\begin{aligned} &= \sum_{m=1}^n m u_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + \sum_{m=1}^n m u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) - \\ &\quad - \sum_{m=1}^n m u_m \frac{\partial}{\partial u_{m-1}} D_x^n(f(u)) + D_x D_x^n(f(u)) = \end{aligned}$$

by the definition of  $D_x$  and the subscript shift in the latter sum,

$$\begin{aligned} &= \sum_{m=1}^n m u_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + \sum_{m=0}^n (m+1) u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) - \\ &\quad - \sum_{m=0}^{n-1} (m+1) u_{m+1} \frac{\partial}{\partial u_m} D_x^n(f(u)) = \end{aligned}$$

since almost all summands in the latter two sums coincide,

$$= \sum_{m=1}^n m u_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + (n+1) u_{n+1} \frac{\partial}{\partial u_n} D_x^n(f(u)) =$$

by (17),

$$\begin{aligned} &= \sum_{m=1}^n mu_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)) + (n+1)u_{n+1} \frac{\partial}{\partial u_{n+1}} D_x^{n+1}(f(u)) = \\ &= \sum_{m=1}^{n+1} mu_m \frac{\partial}{\partial u_m} D_x^{n+1}(f(u)). \quad \text{Q. E. D.} \end{aligned}$$

□

**Lemma 4.**  $[\hat{X}, U_t]^{\text{FN}} \lrcorner du_k = [\hat{X}, U_t]^{\text{FN}} \lrcorner du_{\bar{k}} = 0, k \geq 1.$

*Proof.* Let  $k$  be a positive integer. Taking (15) into account, consider the 1-form

$$[\hat{X}, U_t]^{\text{FN}} \lrcorner du_k = \left( (k-1)\bar{D}_y u_k - \sum_{l=1}^{k-1} lu_l \frac{\partial}{\partial u_l} \bar{D}_x^{k-1}(\exp(2u)) \right) \cdot dy;$$

we also see that all coefficients of  $dx, du, du_l, du_{\bar{l}}$  vanish for all  $l \geq 1$ . Finally, note that  $\bar{D}_y u_k = \bar{D}_x^{k-1}(\exp(2u))$ . By Lemma 3, the coefficient of  $dy$  is trivial. Arguing as above, we see that  $[\hat{X}, U_t]^{\text{FN}} \lrcorner du_{\bar{k}} = 0$ . This completes the proof. □

**Theorem 1.** *The  $\tau_t$ -shadow (10) satisfies the relation*

$$[\hat{X}, U_t]^{\text{FN}} = (\exp(\tilde{u} + u - t) dx - 2 \exp(t) \sinh(\tilde{u} - u) dy) \otimes \frac{\partial}{\partial \tilde{u}},$$

*i.e., the group of diffeomorphisms  $A_t = \exp(t\hat{X})$  provides smooth one-parametric family (5) of non-equivalent one-dimensional non-abelian coverings over Liouville's equation (3). These coverings correspond to Bäcklund autotransformations for the Liouville equation. Bäcklund transformations are given by diagram (2). Infinitesimal part of the structural element is given by (14).*

**Final remark.** The reader will have no difficulty in showing that quite analogous theorems are valid for Bäcklund transformations between Liouville's equation (3) and the wave equation  $v_{xy} = 0$ :

$$(v - u)_x = \exp(-t) \exp(u + v), \quad (v + u)_y = -\exp(t) \exp(u - v), \quad t \in \mathbb{R},$$

and Bäcklund transformations between the Liouville equation and the  $\text{scal}^+$ -Liouville equation  $\Upsilon_{xy} = \exp(-2\Upsilon)$ :

$$(\Upsilon - u)_x = 2 \exp(-t) \cosh(\Upsilon + u), \quad (\Upsilon + u)_y = -\exp(t) \exp(u - \Upsilon), \quad t \in \mathbb{R}.$$

Scaling symmetry (10) is the required  $\tau_t$ -shadow in both cases, and Lemma 3 is an efficient tool to prove these theorems.

Problems of direct integration in nonlocal variables of these Bäcklund (auto)transformations and their permutability will be the object of another paper.

**Acknowledgement.** The author would like to express his gratitude to Prof. I. S. Krasil'shchik for his essential remarks and constructive criticism. The author thanks A. V. Ovchinnikov for improving the readability of this paper.

## REFERENCES

- [1] A.V.Bocharov, V.N.Chetverikov, S.V.Duzhin, N.G.Khor'kova, I.S.Krasil'shchik, A.V.Samokhin, Yu.N.Torkhov, A.M.Verbovetsky, and A.M.Vinogradov, *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics* (Amer. Math. Soc., Providence, RI, 1999). Edited and with a preface by Krasil'shchik and Vinogradov.
- [2] S.Igonin and I.S.Krasil'shchik, *Advanced Studies in Pure Mathematics*, Mathematical Society of Japan, (2001) (to appear). See also [arXiv:nlin.SI/0010040](https://arxiv.org/abs/nlin.SI/0010040).
- [3] I.S. Krasil'shchik and P.H.M. Kersten *Symmetries and recursion operators for classical and supersymmetric differential equations* (Dordrecht etc.: Kluwer Acad. Publ., 2000).
- [4] R.K.Dodd and R.K.Bullough *Proc. Roy. Soc., London.* **A351** (1976).

LOMONOSOV MOSCOW STATE UNIVERSITY, FACULTY OF PHYSICS, DEPARTMENT OF MATHEMATICS, MOSCOW, RUSSIA., INDEPENDENT UNIVERSITY OF MOSCOW, MATH COLLEGE, MOSCOW, RUSSIA.

*E-mail address:* arthemymccme.ru