The Diffiety Institute Preprint Series

Preprint DIPS–01 **January 24, 2002**

Flatness conditions for control systems

by

V. N. CHETVERIKOV

Available via INTERNET:
http://diffiety.ac.ru/

The Diffiety Institute
Polevaya 6-45, Pereslavl-Zalessky, 152140 Russia.

Flatness conditions for control systems

V. N. CHETVERIKOV

Abstract. We study the problem of looking for flat control systems and flat outputs. A control system is flat if there exists a change of variables transforming this system to the trivial system: $\dot{x} = u$. The change of variables can be of most general type. In this work, a necessary and sufficient condition for flatness is obtained. We give also a flatness test and a method for constructing the change of variables. An example is considered to illustrate the test and the method.

1. Introduction

Flat control systems were introduced by Fliess, Lévine, Martin, and Rouchon. Later on, numerous interesting examples of flat control systems was found (see, for example, [3, 4]). Since flatness is equivalent to endogenous dynamic feedback linearization [1, 3], problems of control theory are solvable for flat systems (see [4]). This facts explain the interest in looking for flatness conditions [1].

Using geometric methods, Aranda-Bricaire, Moog, and Pomet introduced an infinitesimal Brunovsky form for nonlinear systems and obtained a necessary and sufficient condition for flatness [1]. Namely flatness of a control system means existence of an invertible differential operator of some type satisfying some conditions (see Theorem 2 below). In geometry of differential equations [6, 7] the operators of this type are investigated and are called $\mathcal{C}$-differential.

In this work, we regard a dynamical control system as an underdetermined system of ordinary differential equations and use a geometric approach to nonlinear differential equations developed by Vinogradov and his school [6, 7]. Main objects of this approach are diffieties. In the case of control systems, the diffieties are infinite dimensional manifolds endowed with one-dimensional distributions, which are called the Cartan distributions.

Besides, we extend the deformation theory of pseudogroup structures defined on finite dimensional manifolds to our infinite dimensional case. This theory was developed by Spencer [9], Guillemin and Sternberg [5] and solves the following problem. Recall that a pseudogroup is a transformation group whose elements are solutions of some partial differential equation. Let some transitive continuous pseudogroup act on a manifold M. Consider some geometric structure on M. This structure is transformated under the action of the pseudogroup. The deformations (transformations) of the structure under the action of the pseudogroup are investigated in this theory. In our case, the manifold M is the diffiety for the control system under consideration. The geometric structure is

Key words and phrases. nonlinear control systems, flatness, $\mathcal{C}$-transformation, dynamic feedback linearization.

This work was supported in part by RFBR grant 99-01-00863.

a vector version of the complex of Cartan forms (9). The pseudogroup is the group of changes of variables on the diffiety.

Using properties of $\mathcal{C}$-differential operators and ideas of the deformation theory, we rewrite the condition from [1] and get new flatness conditions (see Theorem 4 below). These conditions are equations on some form-valued differential operator R. To find a flat output it is necessary and sufficient to solve these equations.

The paper is organized as follows. In section 2, we present a geometric framework for control systems, like [4, 8]. In section 3, results from [1] are generalized to the nonautonomous case. Section 4 contains a result about higher symmetries of control systems. In geometry of differential equations higher symmetries of an equation are interpreted as vector fields on a space of solutions of the equation. We use higher symmetries in section 5, where we give our main result (Theorem 4) and a method for search of flat outputs. In section 6, we solve our flatness conditions by means of a small parameter method exposed in [5]. An example of application of our methods is also discussed there.

2. A geometric framework

Consider a smooth control system

$$\dot{x} = f(t, x, u), \tag{1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is the state, $u = (u_1, \dots, u_m) \in \mathbb{R}^m$ is the control input. For system (1), the *diffiety* (or *infinitely prolonged system*) is an infinite dimensional manifold $\mathcal{E}^\infty$ with coordinates

$$(t, x, u^{(0)}, u^{(1)}, \dots, u^{(l)}, \dots), \tag{2}$$

where $u^{(l)}$ denotes the vector variable corresponding to the lth order derivative of u with respect to t. Every solution $s = (s_x(t), s_u(t))$ of system (1) defined in a neighborhood of a point t_0 determines a point of $\mathcal{E}^\infty$ with the coordinates

$$t = t_0, \qquad x = s_x(t_0), \qquad u^{(l)} = \frac{\partial^l s_u(t_0)}{\partial t^l}, \qquad l \geq 0.$$

This point is called the *infinite jet of the solution s at the point t_0*. From the existence theorem for systems of ordinary differential equations it follows that any point of the diffiety $\mathcal{E}^\infty$ is an infinite jet of some solution of (1).

The *Cartan distribution* on $\mathcal{E}^\infty$ is one-dimensional and is determined by the vector field

$$D = \frac{\partial}{\partial t} + f(t, x, u^{(0)})\frac{\partial}{\partial x} + u^{(1)}\frac{\partial}{\partial u^{(0)}} + \dots + u^{(s+1)}\frac{\partial}{\partial u^{(s)}} + \dots, \tag{3}$$

which is called the *total derivative with respect to t*. The Lie derivative along D is simply the time-derivative according to system (1). We denote by $D\omega$ the Lie derivative of a differential form (or a function) ω along D.

A smooth function on $\mathcal{E}^\infty$ is a function smoothly depending on a finite (but arbitrary) number of coordinates (2). By $\mathcal{F}(\mathcal{E})$ denote the $\mathbb{R}$-algebra of smooth functions on $\mathcal{E}^\infty$. Differential forms on $\mathcal{E}^\infty$ are finite sums, whereas vector fields may be given by infinite sums with coefficients in $\mathcal{F}(\mathcal{E})$ (see, for example, (3)).

Let $\mathcal{E}^\infty$ and $\mathcal{S}^\infty$ be two diffieties, $F: \mathcal{E}^\infty \longrightarrow \mathcal{S}^\infty$ a diffeomorphism preserving the independent variable, i. e., $F^*(t) = t$. The diffeomorphism F is called a *$\mathcal{C}$-transformation* [6], a *Lie–Bäcklund isomorphism* [3] or an *equivalence by endogenous dynamic feedback* [1, 3] if it preserves the Cartan distribution, i. e., $F_*(D) = D$. In this definition the diffieties $\mathcal{E}^\infty$ and $\mathcal{S}^\infty$ can be changed to their open subsets.

Let $\mathcal{E}^\infty$ be the diffiety for system (1), $\theta \in \mathcal{E}^\infty$. System (1) is said to be *flat* [3] or *endogenous dynamic feedback linearizable* [1, 3] around θ if there exists a $\mathcal{C}$-transformation F of a neighborhood of θ to an open subset of the diffiety for the *trivial system*

$$\dot{x} = u, \quad x \in \mathbb{R}^m, \quad u \in \mathbb{R}^m.$$

In what follows, $\mathcal{E}^\infty$ denotes the diffiety for system (1).

3. B-basises and B-regular points

Here we remind some concepts from [1] and simultaneously generalize them to the nonautonomous case.

Let $\mathcal{C}^1\Lambda(\mathcal{E})$ be the $\mathcal{F}(\mathcal{E})$-module of differential 1-forms annihilating the Cartan distribution, i. e.,

$$\omega \in \mathcal{C}^1\Lambda(\mathcal{E}) \quad \Longleftrightarrow \quad \omega(D) = 0.$$

Define the operator $d_\mathcal{C}: \mathcal{F}(\mathcal{E}) \longrightarrow \mathcal{C}^1\Lambda(\mathcal{E})$ by the rule $g \mapsto dg - D(g)dt$. The operator $d_\mathcal{C}$ possesses many properties of the de Rham differential d. In particular,

$$d_\mathcal{C} g(x, u^{(0)}, \dots) = \sum_i \frac{\partial g}{\partial x_i} d_\mathcal{C} x_i + \sum_j \frac{\partial g}{\partial u_j^{(0)}} d_\mathcal{C} u_j^{(0)} + \dots.$$

However $d_\mathcal{C} g = 0$ iff g is a function of t.

Obviously, in coordinate system (2) the module $\mathcal{C}^1\Lambda(\mathcal{E})$ is generated by forms

$$d_\mathcal{C} x_1, \dots, d_\mathcal{C} x_n, d_\mathcal{C} u_1^{(0)}, \dots, d_\mathcal{C} u_m^{(0)}, \dots, d_\mathcal{C} u_1^{(l)}, \dots, d_\mathcal{C} u_m^{(l)}, \dots.$$

Denote by $\mathcal{H}_0$ the $\mathcal{F}(\mathcal{E})$-submodule of $\mathcal{C}^1\Lambda(\mathcal{E})$ generated by forms $d_\mathcal{C} x_1, \dots, d_\mathcal{C} x_n$. By definition, put

$$\mathcal{H}_{k+1} = \{\omega \in \mathcal{H}_k \,|\, D\omega \in \mathcal{H}_k\}, \quad k \geq 0.$$

The *dimension* of some submodule $\mathcal{H} \subset \mathcal{C}^1\Lambda(\mathcal{E})$ at a point $\theta \in \mathcal{E}^\infty$ is the dimension of the space of covectors $\{\omega_\theta \,|\, \omega \in \mathcal{H}\}$. A point $\theta \in \mathcal{E}^\infty$ is called *Brunovský-regular* (or shortly *B-regular*) if in a neighborhood of θ the rank of the matrix $\partial f / \partial u$ is equal to m and the dimensions of $\mathcal{H}_k$ and $\mathcal{H}_k + D(\mathcal{H}_k)$ are constant for any $k > 0$.

Note that the dimension of $\mathcal{H}_k$ at any point is finite and $\mathcal{H}_{k+1} \subset \mathcal{H}_k$. It follows that in a neighborhood of a B-regular point there exists an integer k^* such that $\mathcal{H}_{k+1} = \mathcal{H}_k = \mathcal{H}_{k^*}$ for $k \geq k^*$. By ρ denote the dimension of $\mathcal{H}_{k^*}$ in a neighborhood of a B-regular point under consideration.

Remark 1. In the autonomous case we can consider only functions, differential forms, and vector fields that are independent of t. In this case, $d_\mathcal{C} g = dg$, $\mathcal{E}^\infty$ is a manifold with coordinates $(x, u^{(0)}, \dots, u^{(l)}, \dots)$ (without t), $\mathcal{C}^1\Lambda(\mathcal{E})$ is identified with $\Lambda^1(\mathcal{E}^\infty)$. Also, all concepts and facts from this section are transformed to concepts and facts from [1].

Theorem 1. [1] *In a neighborhood of a B-regular point for system (1) there exist* ρ *functions* $\chi_1, \dots, \chi_\rho$ *of* $t, x_1, \dots, x_n$ *and* m *forms* $\omega_1, \dots, \omega_m \in \mathcal{H}_0$ *such that*
(1) $\{d_\mathcal{C}\chi_1, \dots, d_\mathcal{C}\chi_\rho\}$ *is a basis of the module* $\mathcal{H}_{k^*}$*;*
(2) $\{d_\mathcal{C}\chi_1, \dots, d_\mathcal{C}\chi_\rho\} \cup \{D^j(\omega_k) \,|\, k = 1, \dots, m, j \geq 0\}$ *is a basis of the module* $\mathcal{C}^1\Lambda(\mathcal{E})$*.*

It is easily shown that nonautonomous system (1) has an infinitesimal Brunovský form similar to that given in [1] for the autonomous case.

Further, if system (1) is flat around a B-regular point θ, then $\rho = 0$. In this case, the set $\{\omega_1, \dots, \omega_m\}$ is called a *B-basis* or a *linearizing Pfaffian system* [1] at the point θ. The proof of Theorem 1 is similar to the proof of the corresponding theorem from [1] and gives a **procedure for finding a B-basis**. Namely for $\rho = 0$ we have $\mathcal{H}_{k^*} = 0$ and $\mathcal{H}_{k^*-1} \neq 0$. Choose a basis $\omega_1, \dots, \omega_{m_1}$ $(m_1 \leq m)$ of the module $\mathcal{H}_{k^*-1}$. Then the 1-forms $\omega_1, \dots, \omega_{m_1}, D(\omega_1), \dots, D(\omega_{m_1})$ are $\mathcal{F}(\mathcal{E})$-linear independent and lie in $\mathcal{H}_{k^*-2}$. To obtain a basis of the module $\mathcal{H}_{k^*-2}$ we add some 1-forms $\omega_{m_1+1}, \dots, \omega_{m_2}$ $(m_1 \leq m_2 \leq m)$ to this collection. Repeating this construction for $\mathcal{H}_{k^*-3}, \dots, \mathcal{H}_0$, we get a B-basis $\{\omega_1, \dots, \omega_m\}$.

Denote

$$\mathcal{C}^s\Lambda(\mathcal{E}) = \underbrace{\mathcal{C}^1\Lambda(\mathcal{E}) \wedge \dots \wedge \mathcal{C}^1\Lambda(\mathcal{E})}_{s \text{ times}}, \qquad s > 1,$$

$$\mathcal{C}^0\Lambda(\mathcal{E}) = \mathcal{F}(\mathcal{E}), \qquad \mathcal{C}^*\Lambda(\mathcal{E}) = \bigoplus_{s=0}^{\infty} \mathcal{C}^s\Lambda(\mathcal{E}).$$

The elements of $\mathcal{C}^s\Lambda(\mathcal{E})$ are called *Cartan s-forms* on $\mathcal{E}^\infty$. A differential operator of the type

$$\Delta = g_0 + g_1 D + g_2 D^2 + \dots + g_k D^k, \qquad g_0, g_1, \dots, g_k \in \mathcal{F}(\mathcal{E}), \tag{4}$$

is called a *$\mathcal{C}$-differential operator of order k*. A $\mathcal{C}$-differential operator acts on $\Lambda^*(\mathcal{E}^\infty)$ and takes $\mathcal{C}^*\Lambda(\mathcal{E})$ to itself.

Remark 2. The composition of two $\mathcal{C}$-differential operators is a $\mathcal{C}$-differential operator. The set of all $\mathcal{C}$-differential operators on $\mathcal{E}^\infty$ forms a noncommutative $\mathbb{R}$-algebra. The set $\mathcal{C}^1\Lambda(\mathcal{E})$ is a left module over this algebra. From Theorem 1 it follows that in the case $\rho = 0$, this module is free and a B-basis is its basis.

Let $s = s(t)$ be a (local) solution of (1). Denote by $g|_s$ the restriction of the function $g \in \mathcal{F}(\mathcal{E})$ to the set of infinite jets of the section s. Clearly, $g|_s$ is a function of t. The differential operator

$$g_0\big|_s + g_1\big|_s \frac{d}{dt} + g_2\big|_s \frac{d^2}{dt^2} + \dots + g_k\big|_s \frac{d^k}{dt^k}$$

is called the *restriction* of the $\mathcal{C}$-differential operator (4) to s and is denoted by $\Delta|_s$.

Remark 3. It is easily shown that $\Delta(g)|_s = \Delta|_s(g|_s)$ for any function $g \in \mathcal{F}(\mathcal{E})$ and for any solution s. Moreover, if for some differential operator Δ and for any solution s there exists a differential operator $\Delta|_s$ satisfying this equality, then the operator Δ is a $\mathcal{C}$-differential one [7].

A matrix whose entries are $\mathcal{C}$-differential operators determines in the standard way the operator on the set of columns of differential forms. This operator is called a *matrix*

$\mathcal{C}$*-differential operator*. The *order* of a matrix $\mathcal{C}$-differential operator is the maximal order of its entries.

Theorem 2. [1] *Let* θ *be a B-regular point for system (1),* $\rho = 0$*, and* $\{\omega_1, \dots, \omega_m\}$ *be a B-basis at* θ*. System (1) is flat at* θ *if and only if in a neighborhood of* θ *there exist* m *functions* $h_1, \dots, h_m$ *on* $\mathcal{E}^\infty$ *and an invertible matrix* $\mathcal{C}$*-differential operator* Δ *such that*

$$(5)\qquad \begin{pmatrix} d_{\mathcal{C}} h_1 \\ \vdots \\ d_{\mathcal{C}} h_m \end{pmatrix} = \Delta \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix}.$$

The functions $h_1, \dots, h_m$ from Theorem 2 are called *linearizing* or *flat output* [1, 3].

4. Higher symmetries of control systems

A vector field on $\mathcal{E}^\infty$ without the term $\partial/\partial t$ is called *vertical*. A vertical field X on $\mathcal{E}^\infty$ is called a *higher symmetry* of system (1) if $[X, D] = 0$.

Let $\rho = 0$. From Theorem 1 it follows that in a neighborhood of a B-regular point there exists a matrix $\mathcal{C}$-differential operator P of order k^* such that

$$(6)\qquad \begin{pmatrix} d_{\mathcal{C}} x_1 \\ \vdots \\ d_{\mathcal{C}} u_m \end{pmatrix} = P \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_m \end{pmatrix},$$

where $\{\omega_1, \dots, \omega_m\}$ is a B-basis. By P_i denote ith row of the matrix P, i. e.,

$$d_{\mathcal{C}} x_i = P_i(\omega), \qquad d_{\mathcal{C}} u_j = P_{n+j}(\omega),$$

where ω is the m-column consisting of the forms $\omega_1, \dots, \omega_m$.

Theorem 3. [2] *Suppose* θ *is a B-regular point for system (1) and* $\rho = 0$*. Then in a neighborhood of* θ *any higher symmetry of system (1) has the form*

$$S_\varphi = \sum_{i=1}^{n} P_i(\varphi) \frac{\partial}{\partial x_i} + \sum_{k=0}^{\infty} \sum_{j=1}^{m} D^k \Big(P_{n+j}(\varphi) \Big) \frac{\partial}{\partial u_j^{(k)}},$$

where φ *is a* m*-column of arbitrary functions* $\varphi_1, \dots, \varphi_m \in \mathcal{F}(\mathcal{E})$*.*

The vector function φ is called the *generating function* of the symmetry S_φ. Clearly, the Lie derivative along any symmetry S_φ takes $\mathcal{C}^i\Lambda(\mathcal{E})$ to $\mathcal{C}^i\Lambda(\mathcal{E})$. Below the Lie derivative along S_φ is also denoted by S_φ.

5. Main results

A map A of $\mathcal{C}^*\Lambda(\mathcal{E})$ to $\mathcal{C}^*\Lambda(\mathcal{E})$ such that

$$(7)\qquad A(\Omega) = \alpha_0 \wedge \Omega + \alpha_1 \wedge D\Omega + \cdots + \alpha_k \wedge D^k\Omega, \qquad \alpha_0, \alpha_1, \dots, \alpha_k \in \mathcal{C}^r\Lambda(\mathcal{E}),$$

is called a $\mathcal{C}\Lambda$*-operator of order* k *and of grading* r. The $\mathcal{C}\Lambda$-operator (7) is denoted by

$$(8)\qquad A = \alpha_0 \wedge 1 + \alpha_1 \wedge D + \cdots + \alpha_k \wedge D^k.$$

A matrix whose entries are $\mathcal{C}\Lambda$-operator of order $\le k$ and of grading r is called *matrix* $\mathcal{C}\Lambda$*-operator of order* $\le k$ *and of grading* r. Obviously, matrix $\mathcal{C}$-differential operators of order $\leq k$ are matrix $\mathcal{C}\Lambda$-operators of order $\leq k$ and of grading 0. Matrix $\mathcal{C}\Lambda$-operators are main objects of our approach.

The operator $d_{\mathcal{C}} : \mathcal{C}^0\Lambda(\mathcal{E}) = \mathcal{F}(\mathcal{E}) \longrightarrow \mathcal{C}^1\Lambda(\mathcal{E})$ can be extended to $\Lambda^*(\mathcal{E}^\infty)$ [7]. We use only the extensions to $\mathcal{C}^*\Lambda(\mathcal{E})$ given by

$$d_{\mathcal{C}}\Big(f d_{\mathcal{C}} g_1 \wedge \cdots \wedge d_{\mathcal{C}} g_i\Big) = d_{\mathcal{C}} f \wedge d_{\mathcal{C}} g_1 \wedge \cdots \wedge d_{\mathcal{C}} g_i,$$

where $i > 0$ and $f, g_1, \ldots, g_i \in \mathcal{F}(\mathcal{E})$. Since the module $\mathcal{C}^1\Lambda(\mathcal{E})$ is generated by the image of the operator $d_{\mathcal{C}}$, the map $d_{\mathcal{C}}$ from $\mathcal{C}^i\Lambda(\mathcal{E})$ to $\mathcal{C}^{i+1}\Lambda(\mathcal{E})$ is uniquely defined by the last equality for any $i > 0$. It can be proved that the sequence

$$0 \longrightarrow \mathcal{F}(\mathcal{E}) \xrightarrow{d_{\mathcal{C}}} \mathcal{C}^1\Lambda(\mathcal{E}) \longrightarrow \cdots \longrightarrow \mathcal{C}^i\Lambda(\mathcal{E}) \xrightarrow{d_{\mathcal{C}}} \mathcal{C}^{i+1}\Lambda(\mathcal{E}) \longrightarrow \cdots \tag{9}$$

is exact in all terms except for the term $\mathcal{F}(\mathcal{E})$. The homology in the term $\mathcal{F}(\mathcal{E})$ is equal to the algebra of all functions of t. The map $d_{\mathcal{C}}$ is called the *Cartan differential* on $\mathcal{E}^\infty$.

Further, extend $d_{\mathcal{C}}$ (and S_φ) to columns of forms by the rule $(\omega_i) \mapsto (d_{\mathcal{C}}\omega_i)$. Denote by $[,]$ and $[,]_+$ the commutator and the anticommutator of maps, i. e.,

$$[A, B] = A \circ B - B \circ A, \qquad [A, B]_+ = A \circ B + B \circ A.$$

Note that $d_{\mathcal{C}}$ and S_φ are not $\mathcal{C}\Lambda$-operators. But their commutators or their anticommutators with $\mathcal{C}\Lambda$-operators can be $\mathcal{C}\Lambda$-operators. Indeed, consider the $\mathcal{C}\Lambda$-operator (8) of grading r. Since $d_{\mathcal{C}}$ and D commute, the $\mathcal{C}\Lambda$-operator

$$d_{\mathcal{C}}\alpha_0 \wedge 1 + d_{\mathcal{C}}\alpha_1 \wedge D + \cdots + d_{\mathcal{C}}\alpha_k \wedge D^k$$

is $[d_{\mathcal{C}}, A]$ for even r and $[d_{\mathcal{C}}, A]_+$ for odd r. Hence $[d_{\mathcal{C}}, A]$ for even r and $[d_{\mathcal{C}}, A]_+$ for odd r are $\mathcal{C}\Lambda$-operators of order $\leq k$ and of grading $r+1$. Similarly, $[S_\varphi, A]$ is a $\mathcal{C}\Lambda$-operator of order $\leq k$ and of grading r for any $r \geq 0$. Obviously, the same is valid for a matrix $\mathcal{C}\Lambda$-operator A.

The *internal product* $X\rfloor A$ of a vector field X on $\mathcal{E}^\infty$ and a $\mathcal{C}\Lambda$-operator A of grading r of the form (8) is the following $\mathcal{C}\Lambda$-operator of grading $r-1$:

$$X\rfloor A = (X\rfloor\alpha_0) \wedge 1 + (X\rfloor\alpha_1) \wedge D + \cdots + (X\rfloor\alpha_k) \wedge D^k,$$

where $X\rfloor\alpha$ denotes the internal product of X and a differential form α. For a matrix $\mathcal{C}\Lambda$-operator R with entries R_{ij}, the matrix $\mathcal{C}\Lambda$-operator $X\rfloor R$ with entries $X\rfloor R_{ij}$ is called the *internal product* of X and R.

Consider a matrix $\mathcal{C}\Lambda$-operator R of grading 1 (see Theorem 4). For any matrix $\mathcal{C}\Lambda$-operator A and any symmetry S_φ, denote

$$\begin{aligned} S_\varphi\{A\} &= \Big[S_\varphi, A\Big] + \Big(S_\varphi\rfloor R\Big) \circ A, \\ S_\varphi\langle A\rangle &= \Big[S_\varphi, A\Big] - A \circ \Big(S_\varphi\rfloor R\Big). \end{aligned}$$

Theorem 4. *Let θ be a B-regular point for system (1), $\rho = 0$, and let ω be the m-column of 1-forms $\omega_1, \ldots, \omega_m$ forming a B-basis at θ. System (1) is flat at θ if and only if in a neighborhood of θ there exists a matrix $\mathcal{C}\Lambda$-operator R of grading 1 such that*
(A) $(d_{\mathcal{C}} + R)(\omega) = 0$;

(B) $[d_{\mathcal{C}}, R]_+ + R \circ R = 0$*;*
(C) the orders of the operators

$$S_{\varphi_k}\Big\{\dots S_{\varphi_2}\{S_{\varphi_1}\rfloor R\}\dots\Big\} \quad \textit{and} \quad S_{\varphi_k}\Big\langle\dots S_{\varphi_2}\langle S_{\varphi_1}\rfloor R\rangle\dots\Big\rangle$$

are limited for any positive integer k and any collection of higher symmetries $S_{\varphi_1}, \dots, S_{\varphi_k}$.

The condition (B) from the theorem is a well-known equation from the deformation theory (see [9] and [5]). The condition (C) is a typical one for the infinite-dimensional case and allows us to pass to the finite dimensional case. Any nilpotent operator R satisfies the condition (C).

Proof of necessity. By Theorem 2, there exist an invertible matrix $\mathcal{C}$-differential operator Δ and a m-column h of the functions $h_1, \dots, h_m$ satisfying (5). Set

$$R = \Delta^{-1} \circ [d_{\mathcal{C}}, \Delta]. \tag{10}$$

Obviously,

$$d_{\mathcal{C}} + R = \Delta^{-1} \circ d_{\mathcal{C}} \circ \Delta. \tag{11}$$

If we combine this with ${d_{\mathcal{C}}}^2 = 0$ and $\Delta(\omega) = d_{\mathcal{C}} h$, we get (A) and (B):

$$(d_{\mathcal{C}} + R)(\omega) = (\Delta^{-1} \circ d_{\mathcal{C}} \circ \Delta)(\omega) = (\Delta^{-1} \circ d_{\mathcal{C}} \circ d_{\mathcal{C}})(h) = 0,$$
$$[d_{\mathcal{C}}, R]_+ + R \circ R = (d_{\mathcal{C}} + R)^2 = \Delta^{-1} \circ d_{\mathcal{C}} \circ \Delta \circ \Delta^{-1} \circ d_{\mathcal{C}} \circ \Delta = 0.$$

Since S_φ and D commute, we see that

$$S_\varphi\rfloor R = S_\varphi\rfloor(\Delta^{-1} \circ [d_{\mathcal{C}}, \Delta]) = \Delta^{-1} \circ [S_\varphi, \Delta]. \tag{12}$$

It now follows that the order of the operator $S_\varphi\rfloor R$ is less than or equal to the sum of the orders of Δ^{-1} and of Δ. Using (12), we get also

$$\Big[S_{\varphi_2}, [S_{\varphi_1}, \Delta]\Big] = \Big[S_{\varphi_2}, \Delta \circ (S_{\varphi_1}\rfloor R)\Big] = [S_{\varphi_2}, \Delta] \circ (S_{\varphi_1}\rfloor R) + \Delta \circ [S_{\varphi_2}, S_{\varphi_1}\rfloor R]$$
$$= \Delta \circ \Big((S_{\varphi_2}\rfloor R) \circ (S_{\varphi_1}\rfloor R) + [S_{\varphi_2}, S_{\varphi_1}\rfloor R]\Big) = \Delta \circ S_{\varphi_2}\{S_{\varphi_1}\rfloor R\}.$$

As above, the order of the operator $S_{\varphi_2}\{S_{\varphi_1}\rfloor R\}$ is less than or equal to the same sum of the orders.

On the other hand,

$$0 = [d_{\mathcal{C}}, \Delta^{-1} \circ \Delta] = [d_{\mathcal{C}}, \Delta^{-1}] \circ \Delta + \Delta^{-1} \circ [d_{\mathcal{C}}, \Delta].$$

Using this equality, we get consecutively

$$[d_{\mathcal{C}}, \Delta^{-1}] = -R \circ \Delta^{-1},$$
$$[S_\varphi, \Delta^{-1}] = S_\varphi\rfloor[d_{\mathcal{C}}, \Delta^{-1}] = -(S_\varphi\rfloor R) \circ \Delta^{-1},$$
$$\Big[S_{\varphi_2}, [S_{\varphi_1}, \Delta^{-1}]\Big] = -\Big[S_{\varphi_2}, (S_{\varphi_1}\rfloor R) \circ \Delta^{-1}\Big] = -\Big[S_{\varphi_2}, S_{\varphi_1}\rfloor R\Big] \circ \Delta^{-1}$$
$$- (S_{\varphi_1}\rfloor R) \circ \Big[S_{\varphi_2}, \Delta^{-1}\Big] = -S_{\varphi_2}\langle S_{\varphi_1}\rfloor R\rangle \circ \Delta^{-1}.$$

Arguing as above, we see that the order of the operator $S_{\varphi_2}\langle S_{\varphi_1}\rfloor R\rangle$ is also less than or equal to the sum of the orders of Δ^{-1} and of Δ. Continuing this line of reasoning, we get (C). ▶

To prove the sufficiency, we need two lemmas.

Lemma 5. *Let an infinite jet $\theta \in \mathcal{E}^\infty$ of a local solution s of (1) be a B-regular point. Suppose $\rho = 0$, ω is a B-basis at θ, and a matrix $\mathcal{C}\Lambda$-operator R of grading 1 satisfies the conditions (A), (B), and (C) in a neighborhood of θ. Then in a neighborhood of θ there exist unique matrix $\mathcal{C}\Lambda$-operators Δ and ∇ of grading 0 satisfying*

$$[d_\mathcal{C}, \Delta] = \Delta \circ R, \qquad \Delta\big|_s = \mathrm{id}, \tag{13}$$

$$[d_\mathcal{C}, \nabla] = -R \circ \nabla, \qquad \nabla\big|_s = \mathrm{id}, \tag{14}$$

where id *denotes the identical operator.*

Lemma 6. *A left-invertible $\mathcal{C}$-differential operator with a square matrix is two-sided invertible.*

The proof of these lemmas are given after the proof of the second part of Theorem 4.

Proof of sufficiency. From the first equalities in (13) and (14) it follows that

$$[d_\mathcal{C}, \Delta \circ \nabla] = [d_\mathcal{C}, \Delta] \circ \nabla + \Delta \circ [d_\mathcal{C}, \nabla] = 0.$$

Using the second equalities in (13) and (14), we get

$$(\Delta \circ \nabla)|_s = \Delta|_s \circ \nabla|_s = \mathrm{id}\,. \tag{15}$$

On the other hand, the homology of the complex (9) in the term $\mathcal{F}(\mathcal{E})$ is the $\mathbb{R}$-algebra of smooth functions of t. Therefore coefficients of the operator $\Delta \circ \nabla$ at $D^l, l \geq 0$, are functions of t only. Combining this with (15), we get $\Delta \circ \nabla = (\Delta \circ \nabla)|_s = \mathrm{id}$. By Lemma 6, the operator Δ is two-sided invertible.

Thus we have (10) and thereby (11). Combining (11) with the condition (A), we get $d_\mathcal{C}(\Delta(\omega)) = 0$. Since the complex (9) is exact in the term $\mathcal{C}^1\Lambda(\mathcal{E})$, we have $\Delta(\omega) = d_\mathcal{C} h$ for some vector function h. By Theorem 2, system (1) is flat. ▶

Proof of Lemma 5. Let the orders of the operators from the condition (C) be limited by an integer L. Consider the set C_R of matrix $\mathcal{C}$-differential operators Δ of dimension $m \times m$ and of order $\leq L$ such that for any positive integer k and any collection of higher symmetries $S_{\varphi_1}, \ldots, S_{\varphi_k}$ the order of the operator $\Delta \circ S_{\varphi_k}\{\ldots S_{\varphi_2}\{S_{\varphi_1} \rfloor R\} \ldots\}$ is less than or equal to L. The set C_R is nonempty, since it contains the identical operator id.

We shall prove that in a neighborhood of the point θ there exists an operator Δ from C_R satisfying (13). Clearly, C_R is a module over $\mathcal{F}(\mathcal{E})$. Besides, the orders of operators from C_R are limited. Therefore the module C_R has a finite basis $\{\Delta_1, \ldots, \Delta_l\}$. Suppose

$$\Delta = \sum_{i=1}^{l} f_i \Delta_i, \qquad f_i \in \mathcal{F}(\mathcal{E}); \tag{16}$$

then equation (13) is written as

$$\sum d_\mathcal{C} f_i \wedge \Delta_i + \sum f_i \Big([d_\mathcal{C}, \Delta_i] - \Delta_i \circ R\Big) = 0. \tag{17}$$

Now we show that $S_\varphi\langle\Delta\rangle \in C_R$ if $\Delta \in C_R$. For any symmetry S_φ and any matrix $\mathcal{C}$-differential operators Δ and $\Box$ one has

$$[S_\varphi, \Delta \circ \Box] = S_\varphi\langle\Delta\rangle \circ \Box + \Delta \circ S_\varphi\{\Box\}. \tag{18}$$

Put $\Box = S_{\varphi_k}\langle \ldots S_{\varphi_2}\langle S_{\varphi_1} \rfloor R\rangle \ldots\rangle$. Since $\Delta \in C_R$, the orders of the operators $\Delta \circ \Box$ and $\Delta \circ S_\varphi\{\Box\}$ are less than or equal to L. Commutation with a symmetry S_φ does not increase order of a $\mathcal{C}$-differential operator. Therefore, the order of $[S_\varphi, \Delta\circ\Box]$ is less than or equal to L. Due to (18), the same is valid for $S_\varphi\langle\Delta\rangle\circ\Box$. That is $S_\varphi\langle\Delta\rangle \in C_R$.

On the other hand, for any symmetry S_φ one has

$$S_\varphi\rfloor\Big([d_{\mathcal{C}}, \Delta_i] - \Delta_i\circ R\Big) = S_\varphi\langle\Delta_i\rangle \in C_R.$$

Since higher symmetries generate $\mathcal{F}(\mathcal{E})$-module of vertical fields, the last inclusion means that

$$[d_{\mathcal{C}}, \Delta_i] - \Delta_i\circ R = \sum_{j=1}^{l}\omega_{ij}\wedge\Delta_j, \qquad i = 1,\dots,l, \tag{19}$$

where $\omega_{ij}\in\mathcal{C}^1\Lambda(\mathcal{E})$ for any i, j.

Combining (17) and (19), we get

$$\sum_{i=1}^{l}\Big(d_{\mathcal{C}}f_i + \sum_{j=1}^{l} f_j\omega_{ji}\Big)\wedge\Delta_i = 0.$$

Since $\{\Delta_1,\dots,\Delta_l\}$ is a basis, we see that equation (13) is equivalent to the system

$$d_{\mathcal{C}}f_i + \sum_{j=1}^{l} f_j\omega_{ji} = 0, \quad i = 1,\dots,l. \tag{20}$$

Note that the forms $\omega_{ji}, D\omega_{ji}, j, i = 1,\dots,l$, depend on a finite number of coordinates (2). Denote by $z_1,\dots,z_s$ the coordinates from the collection (2) without t, on which these forms depend. To prove solvability of system (20), we introduce auxiliary variables $v_1,\dots,v_l$ and consider a manifold M with the coordinates $t, z_1,\dots,z_s, v_1,\dots,$ v_l. Let $\mathcal{P}$ be the distribution determined by the forms

$$dt, \quad dv_i + \sum_{j=1}^{l} v_j\omega_{ji}, \quad i = 1,\dots,l.$$

Our aim now is to prove that the distribution $\mathcal{P}$ is integrable. Since $d\omega_{ji} = d_{\mathcal{C}}\omega_{ji} + dt\wedge D\omega_{ji}$, we have

$$d\Big(dv_i + \sum_{j=1}^{l} v_j\omega_{ji}\Big) = \sum_{j=1}^{l}\Big(dv_j\wedge\omega_{ji} + v_j d_{\mathcal{C}}\omega_{ji} + v_j dt\wedge D\omega_{ji}\Big).$$

Whence,

$$d\Big(dv_i + \sum_{j=1}^{l} v_j\omega_{ji}\Big) = \sum_{j=1}^{l}\Big(dv_j + \sum_{p=1}^{l} v_p\omega_{pj}\Big)\wedge\omega_{ji} + \sum_{j=1}^{l}\Big(v_j dt\wedge D\omega_{ji}\Big) + \sum_{j=1}^{l} v_j\Big(d_{\mathcal{C}}\omega_{ji} - \sum_{p=1}^{l}\omega_{jp}\wedge\omega_{pi}\Big).$$

Since the forms $d_{\mathcal{C}}\omega_{ji} - \sum \omega_{jp} \wedge \omega_{pi}$ contain $d_{\mathcal{C}} z_\alpha$ only and do not contain dv_α and dt, the distribution $\mathcal{P}$ satisfies the Frobenius condition iff

$$\sum_{j=1}^{l} v_j \Big(d_{\mathcal{C}}\omega_{ji} - \sum_{p=1}^{l} \omega_{jp} \wedge \omega_{pi} \Big) = 0, \quad i = 1, \dots, l.$$

Further, the forms $d_{\mathcal{C}}\omega_{ji} - \sum \omega_{jp} \wedge \omega_{pi}$ are independent of v_j. Therefore the last system is equivalent to

$$d_{\mathcal{C}}\omega_{ji} = \sum_{p=1}^{l} \omega_{jp} \wedge \omega_{pi}, \quad j, i = 1, \dots, l. \tag{21}$$

Now let us prove (21). Using the condition (B) and properties of the commutator and the anticommutator, we get

$$\Big[d_{\mathcal{C}}, [d_{\mathcal{C}}, \Delta_i] - \Delta_i \circ R \Big]_+ = -[d_{\mathcal{C}}, \Delta_i] \circ R - \Delta_i \circ [d_{\mathcal{C}}, R]_+ = -\Big([d_{\mathcal{C}}, \Delta_i] - \Delta_i \circ R \Big) \circ R.$$

Combining this with (19), we get

$$\Big[d_{\mathcal{C}}, \sum_j \omega_{ij} \wedge \Delta_j \Big]_+ + \Big(\sum_j \omega_{ij} \wedge \Delta_j \Big) \circ R = 0.$$

It now follows that

$$0 = \sum_j \Big(d_{\mathcal{C}}\omega_{ij} \wedge \Delta_j - \omega_{ij} \wedge ([d_{\mathcal{C}}, \Delta_j] - \Delta_j \circ R) \Big) = \sum_j \Big(d_{\mathcal{C}}\omega_{ij} \wedge \Delta_j - \sum_p \omega_{ij} \wedge \omega_{jp} \wedge \Delta_p \Big).$$

Since $\{\Delta_1, \dots, \Delta_l\}$ is a basis, we obtain (21).

Thus the distribution $\mathcal{P}$ is integrable. Every maximal integral manifold of $\mathcal{P}$ is given by a system of equations of the form

$$t = \text{const}, \quad v_1 = g_1, \quad \dots, \quad v_l = g_l,$$

where $g_1, \dots, g_l$ are some functions on $\mathcal{E}^\infty$ depending on the coordinates $z_1, \dots, z_s$. This integral manifold is uniquely determined by a value of t and values of the functions $g_1, \dots, g_l$ at a point of $\mathcal{E}^\infty$.

Let the given point $\theta \in \mathcal{E}^\infty$ be the infinite jet of the solution s at a point t_0. Any solution $f_1, \dots, f_l \in \mathcal{F}(\mathcal{E})$ of system (20) corresponds to the collection of maximal integral manifold of $\mathcal{P}$ of the form

$$t = t_1, \quad v_1 = f_1\big|_{t=t_1}, \quad \dots, \quad v_l = f_l\big|_{t=t_1}$$

for t_1 from some neighborhood of the point t_0. So there exists a unique solution of system (20) satisfying the initial conditions

$$f_i\big|_s = e_i\big|_s, \qquad i = 1, \dots, l,$$

where $\sum_i e_i \Delta_i$ is the identical operator id. This means that there exists a unique matrix $\mathcal{C}$-differential operators Δ such that conditions (13) hold.

In the same way, it can be proved existence of the operator ∇ satisfying conditions (14). In this case, it should be replaced the module C_R by the module of matrix $\mathcal{C}$-differential operators ∇ of order $\leq L$ such that for any positive integer k and any collection of higher symmetries $S_{\varphi_1}, \dots, S_{\varphi_k}$ the order of the operator $S_{\varphi_k}\langle \dots S_{\varphi_2}\langle S_{\varphi_1} \rfloor R \rangle \dots \rangle \circ \nabla$ is less than or equal to L. ▶

Proof of Lemma 6. Let Δ and ∇ be $\mathcal{C}$-differential operators with square matrixes and

$$\Delta \circ \nabla = \mathrm{id}\,. \tag{22}$$

Denote by Q the matrix $\mathcal{C}$-differential operator $\nabla \circ \Delta - \mathrm{id}$. To prove that the operator Q vanishes, we introduce some concepts.

Any matrix $\mathcal{C}$-differential operator P can be presented in the form

$$P = \sum_{i=0}^{s} P_i D^i,$$

where s is the order of P and $P_0, P_1, \ldots, P_s$ are matrixes of functions on $\mathcal{E}^\infty$. The matrix $P_s \neq 0$ is called the *symbol* of the operator P and is denoted by $\operatorname{smbl} P$. Clearly, the composition of the operators possesses the following property:

$$P \circ q = (\operatorname{smbl} P \operatorname{smbl} q)\, D^{s+l} + \ldots, \tag{23}$$

where q is another matrix $\mathcal{C}$-differential operator of the corresponding dimension, l is its order, the symbols of the operators P and q are multiplied as matrixes, and the dots denote the terms with $D^i, i < s + l$.

Further note that m-columns of $\mathcal{C}$-differential operators determine in the standard way operators from $\mathcal{F}(\mathcal{E})$ to the set of m-columns of functions from $\mathcal{F}(\mathcal{E})$. Denote by $\mathcal{C}\mathrm{Diff}_l^m$ the set of the m-columns whose entries are $\mathcal{C}$-differential operators of order $\leq l$. This set is a module over $\mathcal{F}(\mathcal{E})$ under the following multiplication. The product of an operator $p \in \mathcal{C}\mathrm{Diff}_l^m$ and a function $f \in \mathcal{F}(\mathcal{E})$ is the operator taking a function $g \in \mathcal{F}(\mathcal{E})$ to the m-column $p(fg)$. Let k be the order of the operator ∇ from (22). Denote by h_∇^l the $\mathcal{F}(\mathcal{E})$-module homomorphism from $\mathcal{C}\mathrm{Diff}_l^m$ to $\mathcal{C}\mathrm{Diff}_{l+k}^m$ taking q to $\nabla \circ q$. From (22) it follows that $\ker h_\nabla^l = 0$ for any $l \geq 0$.

Assume now that $Q \neq 0$. Let $\operatorname{smbl} Q \neq 0$ at a point θ_0 and $r > 0$ be the rang of the matrix $\operatorname{smbl} Q$ at this point. Using (22), we get

$$Q \circ \nabla = \nabla \circ \Delta \circ \nabla - \nabla = 0. \tag{24}$$

This together with formula (23) yields $(\operatorname{smbl} Q \operatorname{smbl} q) = 0$ for any operator $q \in \operatorname{im} h_\nabla^l$ and for any $l \geq 0$. Hence values of symbols of the operators from $\operatorname{im} h_\nabla^l$ at the point θ_0 belong to a space of dimension $n - r$ for any $l \geq 0$. Therefore,

$$\dim \operatorname{im} h_\nabla^l \big|_{\theta_0} \leq (n - r)(l + k + 1), \qquad l \geq 0.$$

On the other hand,

$$\dim \operatorname{im} h_\nabla^l \big|_{\theta_0} = n(l + 1)$$

because $\ker h_\nabla^l \big|_{\theta_0} = 0$. However the inequality $n(l+1) \leq (n-r)(l+k+1)$ cannot hold for sufficiently large l. This contradiction concludes the proof. ▶

Theorems 2 and 4 and Lemma 5 give the following **algorithm for search of flat outputs**.

Step 1: For the control system under consideration, calculate a B-basis (see p. 3), B-regular points, and the operator P (see (6)).

Step 2: Find a matrix $\mathcal{C}\Lambda$-operator R of grading 1 satisfying the conditions (A), (B), and (C).

Step 3: Solving equation (13), one obtains Δ. Here $s = (x(t), u(t))$ is some solution of the control system. To calculate Δ one has no need to know the functions $x(t)$ and $u(t)$ (see the example below).

Step 4: From equation (5) find a flat output.

6. Small parameter method

In this section we discuss step 2 of the algorithm given above. Suppose there exists a parameter ε such that the desired $\mathcal{C}\Lambda$-operator R can be expanded in powers of ε. The parameter ε can be introduced in the following way. Consider a coordinate system on $\mathcal{E}^\infty$

$$(t, z_1, \dots, z_k, y_1, \dots, y_i, \dots) \tag{25}$$

such that the total derivative has the form

$$D = \frac{\partial}{\partial t} + \sum_{i=1}^{k} f_i(t, z, y) \frac{\partial}{\partial z_i} + \sum_{j \geq 1} y_{m+j} \frac{\partial}{\partial y_j}, \tag{26}$$

where $f_1(t, z, y), \dots, f_k(t, z, y)$ are analytic functions in the variables $y_1, y_2, \dots$. Assume also that a B-basis ω and the desired $\mathcal{C}\Lambda$-operator R are analytic in y.

Let us replace y_j by εy_j in D, ω, R for any $j \geq 1$, where ε is a constant. By construction, D, ω, R are analytic in ε. Since $d_{\mathcal{C}} = d - dt \wedge D$, the differential $d_{\mathcal{C}}$ is also analytic in ε. Let

$$\omega = \sum_{i \geq 0} \varepsilon^i \omega_{(i)}, \qquad D = \sum_{i \geq 0} \varepsilon^i D_{(i)}, \qquad d_{\mathcal{C}} = \sum_{i \geq 0} \varepsilon^i d_{(i)}.$$

Expanding the operator R in powers of D and ε, we obtain

$$R = \sum_{i \geq 0} \varepsilon^i R_{(i)}, \qquad R_{(i)} = \sum_{j \geq 0} R_{ij} D^j, \tag{27}$$

where the matrix R_{ij} is independent of ε for any i, j. Notice that $R_{(i)}$ depends on ε because D depends on ε. But the operator $R_{(i)}$ is uniquely determined by the operator

$$R^0_{(i)} = \sum_{j \geq 0} R_{ij} D^j_{(0)},$$

which is independent of ε.

It follows easily that

$$D_{(0)} = \frac{\partial}{\partial t} + \sum_{i=1}^{k} f_i(t, z, 0) \frac{\partial}{\partial z_i} + \sum_{j \geq 1} y_{m+j} \frac{\partial}{\partial y_j}. \tag{28}$$

Besides, the matrixes $R_{0j}, j \geq 0$, are independent of $y_1, y_2, \dots$ and do not contain $d_{\mathcal{C}} y_1, d_{\mathcal{C}} y_2, \dots$. More precisely, $R_{ij}, j \geq 0$, are homogeneous polynomials of degree i in $y_1, y_2, \dots, d_{\mathcal{C}} y_1, d_{\mathcal{C}} y_2, \dots$.

Consider the dynamical system

$$\{\dot{z}_i = f_i(t, z, 0), \qquad i = 1, \dots, k, \tag{29}$$

understood as a control system with the control input $(y_1, \dots, y_m)$. Denote by $\mathcal{E}_0^\infty$ the corresponding diffiety. A coordinate system on $\mathcal{E}_0^\infty$ is (25). The total derivative on $\mathcal{E}_0^\infty$

is vector field (28). Thus the diffieties $\mathcal{E}^\infty$ and $\mathcal{E}_0^\infty$ have the same infinite dimensional manifold, but different Cartan distributions.

Further, since $d_{(0)} = d - dt \wedge D_{(0)}$ is the Cartan differential on $\mathcal{E}_0^\infty$, the sequence

$$0 \longrightarrow \mathcal{F}(\mathcal{E}_0) \xrightarrow{d_{(0)}} \mathcal{C}^1\Lambda(\mathcal{E}_0) \longrightarrow \dots \longrightarrow \mathcal{C}^i\Lambda(\mathcal{E}_0) \xrightarrow{d_{(0)}} \mathcal{C}^{i+1}\Lambda(\mathcal{E}_0) \longrightarrow \dots \tag{30}$$

coincides with sequence (9) for system (29) and consequently is exact in all terms except for the term $\mathcal{F}(\mathcal{E})$.

Finally note that the equalities (A) and (B) are analytic in ε. Expanding differential forms from (A) and operators from (B) in powers of ε, we obtain the collection of equations on operators $R^0_{(i)}$, $i = 0, 1, \dots$. These equations can be solved inductively. Namely in the case $i = 0$, we obtain

$$(d_{(0)} + R^0_{(0)})(\omega_{(0)}) = 0, \qquad [d_{(0)}, R^0_{(0)}]_+ + R^0_{(0)} \circ R^0_{(0)} = 0. \tag{31}$$

Assume that we know $R_j = R_{(0)} + \varepsilon R_{(1)} + \dots + \varepsilon^j R_{(j)}$. From (A) and (B) it follows that

$$(d_{\mathcal{C}} + R_j)(\omega) = \varepsilon^{j+1}\Omega_{(j+1)} + \dots, \tag{32}$$

$$[d_{\mathcal{C}}, R_j]_+ + R_j \circ R_j = \varepsilon^{j+1}Q_{(j+1)} + \dots, \tag{33}$$

where the 2-form $\Omega_{(j+1)}$ and the operator $Q_{(j+1)}$ are independent of ε, the dots denote the terms with ε^s, $s > j+1$. Taking into account coefficients at ε^{j+1} in (A) and (B), we obtain

$$R^0_{(j+1)}(\omega_{(0)}) = -\Omega_{(j+1)}, \qquad [d_{(0)} + R^0_{(0)}, R^0_{(j+1)}]_+ = -Q_{(j+1)}. \tag{34}$$

Theorem 7. *Let the total derivative D for system (1) be of the form (26).*
(1) If system (1) is flat at a B-regular point θ and all the functions $y_1, y_2, \dots$ vanish at θ, then in a neighborhood of θ there exists an invertible matrix $\mathcal{C}$-differential operator Δ_0 on $\mathcal{E}_0^\infty$ such that the $\mathcal{C}\Lambda$-operator on $\mathcal{E}_0^\infty$

$$R^0_{(0)} = \Delta_0^{-1} \circ [d_{(0)}, \Delta_0] \tag{35}$$

satisfies (31).
(2) If the $\mathcal{C}\Lambda$-operator $R^0_{(0)}$ of the form (35) satisfies (31), then for any $j \geq 0$ system (34) is solvable with respect to $R^0_{(j+1)}$.

Proof. Let Δ be the invertible matrix $\mathcal{C}$-differential operator from Theorem 2. As above, we introduce the parameter ε. From the well-known Hadamard lemma it follows that any smooth function g depending on ε can be represented in the form $g = g_0 + \varepsilon g_1$, where g_0 and g_1 are smooth functions, g_0 is independent of ε. This yields that

$$\Delta = \Delta_0 + \varepsilon\Delta_f, \qquad \Delta^{-1} = \nabla_0 + \varepsilon\nabla_f, \tag{36}$$

where $\Delta_0, \Delta_f, \nabla_0$, and ∇_f are smooth differential operators, Δ_0, ∇_0 are independent of ε. Obviously, Δ_0 and ∇_0 are matrix $\mathcal{C}$-differential operators on $\mathcal{E}_0^\infty$. Since

$$\Delta \circ \Delta^{-1} = \Delta_0 \circ \nabla_0 + \varepsilon(\Delta_f \circ \nabla_0 + \Delta_0 \circ \nabla_f + \varepsilon\Delta_f \circ \nabla_f) = \mathrm{id},$$

we see that $\Delta_0 \circ \nabla_0 = \mathrm{id}$. Similarly, $\nabla_0 \circ \Delta_0 = \mathrm{id}$. Therefore, $\nabla_0 = \Delta_0^{-1}$.

Consider now the $\mathcal{C}\Lambda$-operator (10) on $\mathcal{E}^\infty$. The operator $R^0_{(0)}$ defined on page 12 satisfies $R = R^0_{(0)} + \varepsilon R_f$ for some differential operator R_f. Setting $\varepsilon = 0$ in (10), we get (35). In the same way, from the conditions (A) and (B) we get (31). This proves the first statement of the theorem.

To prove the second one, we first show that $\Omega_{(j+1)}$ in (32) is a column of Cartan 2-forms on $\mathcal{E}_0^\infty$ and $Q_{(j+1)}$ in (33) is a $\mathcal{C}\Lambda$-operator of grading 2 on $\mathcal{E}_0^\infty$. It is readily seen that

$$\alpha \in \mathcal{C}^i\Lambda(\mathcal{E}) \quad \Longleftrightarrow \quad D\rfloor\alpha = 0,$$

where $D\rfloor\alpha$ denotes the internal product of D and α. Since the left-hand side of equality (32) is a column of Cartan 2-forms on $\mathcal{E}^\infty$, we have

$$\varepsilon^{j+1} D\rfloor\Omega_{(j+1)} + \cdots = 0.$$

Taking into account coefficients at ε^{j+1}, we obtain $D_{(0)}\rfloor\Omega_{(j+1)} = 0$. This yields that $\Omega_{(j+1)}$ in (32) is a column of Cartan forms on $\mathcal{E}_0^\infty$.

Similarly, the left-hand side of equality (33) is a $\mathcal{C}\Lambda$-operator of grading 2 on $\mathcal{E}^\infty$. Therefore its coefficients at $D^l, l \geq 0$, are matrixes of Cartan 2-forms on $\mathcal{E}^\infty$. Let us expand these forms in powers of ε. As above, the coefficients at ε^{j+1} are Cartan forms on $\mathcal{E}_0^\infty$. To obtain $Q_{(j+1)}$, one needs also to expand the vector field D in powers of ε. Evidently, the operator $Q_{(j+1)}$ contains $D_{(0)}$ and the mentioned Cartan forms on $\mathcal{E}_0^\infty$ only. Therefore it is a $\mathcal{C}\Lambda$-operator on $\mathcal{E}_0^\infty$.

Further, we rewrite (35) as

$$d_{(0)} + R^0_{(0)} = \Delta_0^{-1} \circ d_{(0)} \circ \Delta_0. \tag{37}$$

Using this equality, we can represent (34) as

$$\tilde{R}(\tilde{\omega}) = -\tilde{\Omega}, \tag{38}$$

$$[d_{(0)}, \tilde{R}]_+ = -\tilde{Q}, \tag{39}$$

where

$$\tilde{R} = \Delta_0 \circ R^0_{(j+1)} \circ \Delta_0^{-1}, \qquad \tilde{Q} = \Delta_0 \circ Q^0_{(j+1)} \circ \Delta_0^{-1}$$

are $\mathcal{C}\Lambda$-operators on $\mathcal{E}_0^\infty$,

$$\tilde{\omega} = \Delta_0(\omega_{(0)}), \qquad \tilde{\Omega} = \Delta_0(\Omega_{(j+1)})$$

are columns of Cartan forms on $\mathcal{E}_0^\infty$. Besides, from the first equation in (31) it follows that $d_{(0)}(\tilde{\omega}) = 0$.

Let us expand $\mathcal{C}\Lambda$-operators $\tilde{Q}$ and $\tilde{R}$ in powers of $D_{(0)}$, i. e.,

$$\tilde{Q} = \sum_{\alpha \geq 0} B_\alpha D^\alpha_{(0)}, \qquad \tilde{R} = \sum_{\alpha \geq 0} A_\alpha D^\alpha_{(0)},$$

where entries of matrixes B_α and A_α are Cartan forms on $\mathcal{E}_0^\infty$. Since $d_{(0)}$ and $D_{(0)}$ commute, equation (38) can be written as the system

$$d_{(0)}(A_\alpha) = -B_\alpha \qquad \forall \alpha, \tag{40}$$

where entries of the matrix $d_{(0)}(A_\alpha)$ are the images of the corresponding entries of the matrix A_α under the differential $d_{(0)}$. Formula (39) means that the matrix B_α consists of $d_{(0)}$-exact forms for any α. Since complex (30) is exact in the term $\mathcal{C}^2\Lambda(\mathcal{E}_0)$, system (39) is solvable if and only if

$$d_{(0)}(B_\alpha) = 0 \qquad \forall \alpha.$$

In other notation,

(41) $$[d_{(0)}, \tilde{Q}] = 0.$$

To prove (40), note that

(42) $$[d_{\mathcal{C}} + R_j, (d_{\mathcal{C}} + R_j)^2] = 0.$$

Since ${d_{\mathcal{C}}}^2 = 0$, the operator $(d_{\mathcal{C}} + R_j)^2$ coincides with the left-hand side of equality (33). Combining this with (33) and (41), we get

$$\varepsilon^{j+1}[d_{\mathcal{C}} + R_j, Q_{(j+1)}] + \cdots = 0.$$

Taking into account coefficients at ε^{j+1}, we obtain $[d_{(0)} + R^0_{(0)}, Q_{(j+1)}] = 0$ and consequently (40).

Thus system (39) is solvable. Let $\{\bar{A}_\alpha\}$ be some solution of (39). Then another solution $\{A_\alpha\}$ satisfies

$$d_{(0)}(A_\alpha - \bar{A}_\alpha) = 0 \qquad \forall \alpha.$$

Since complex (30) is also exact in the term $\mathcal{C}^1\Lambda(\mathcal{E}_0)$, we get

$$A_\alpha = \bar{A}_\alpha + d_{(0)}(C_\alpha)$$

for any α and some matrixes C_α of functions on $\mathcal{E}_0^\infty$. Consequently equation (38) is also solvable and any solution of (38) can be represented as

(43) $$\tilde{R} = \bar{R} + [d_{\mathcal{C}}, \Box],$$

where $\bar{R} = \sum_\alpha \bar{A}_\alpha D^\alpha_{(0)}$ is some fixed solution of (38) and $\Box = \sum_\alpha C_\alpha D^\alpha_{(0)}$ is an arbitrary matrix $\mathcal{C}\Lambda$-operator of grading 0 on $\mathcal{E}_0^\infty$.

Now we show that there exists an operator $\Box$ such that the operator (42) satisfies (37), i. e.,

$$(\bar{R} + [d_{(0)}, \Box])(\tilde{\omega}) = -\tilde{\Omega}.$$

Since $d_{(0)}(\tilde{\omega}) = 0$, the last equality is equivalent to

(44) $$d_{(0)}\big(\Box(\tilde{\omega})\big) = -\bar{R}(\tilde{\omega}) - \tilde{\Omega}.$$

From (33) and (34) it follows that the operator $(d_{\mathcal{C}} + R_{j+1})^2$ has the terms only with ε^s, $s \geq j+2$. On the other hand, using the definition of R_{j+1} and (32), we get

$$(d_{\mathcal{C}} + R_{j+1})^2(\omega) = (d_{\mathcal{C}} + R_{j+1})(\varepsilon^{j+1}\Omega_{(j+1)} + \cdots + \varepsilon^{j+1}R_{(j+1)}(\omega)).$$

Taking into account coefficients at ε^{j+1}, we obtain

$$(d_{(0)} + R^0_{(0)})\big(\Omega_{(j+1)} + R^0_{(j+1)}(\omega_{(0)})\big) = 0.$$

Combining this with (6), we get

(45) $$d_{(0)}(\bar{R}(\tilde{\omega}) + \tilde{\Omega}) = 0.$$

Since complex (30) is exact in the term $\mathcal{C}^2\Lambda(\mathcal{E}_0)$, we see that

$$\bar{R}(\tilde{\omega}) + \tilde{\Omega} = d_{(0)}\beta_0$$

for some column β_0 of Cartan 1-forms on $\mathcal{E}_0^\infty$.

Substituting $d_{\mathcal{C}}$ for $d_{(0)}$ in β_0, we obtain the column β of Cartan 1-forms on $\mathcal{E}^\infty$ such that $\beta|_{\varepsilon=0} = \beta_0$. Since ω is a B-basis on $\mathcal{E}^\infty$, there exists matrix $\mathcal{C}$-differential operator ∇ on $\mathcal{E}^\infty$ such that $\nabla(\omega) = -\beta$. If we put $\varepsilon = 0$, we get $\nabla|_{\varepsilon=0}(\omega_{(0)}) = -\beta_0$. Arguing as above, we see that $\nabla|_{\varepsilon=0}$ is a matrix $\mathcal{C}$-differential operator on $\mathcal{E}_0^\infty$. To conclude the proof, it remains to note that the matrix $\mathcal{C}$-differential operator $\square = \nabla|_{\varepsilon=0} \circ \Delta_0^{-1}$ satisfies (43). ▶

Using the theorem, we obtain series (27). However this series can diverge and is not unique. If an operator $R^0_{(j+1)}$ satisfies (34) and $\square$ is an arbitrary matrix $\mathcal{C}$-differential operator such that

$$(d_{(0)} + R^0_{(0)})(\square(\omega_{(0)})) = 0,$$

then the operator

$$R^0_{(j+1)} + [d_{(0)} + R^0_{(0)}, \square]$$

also satisfies (34).

When D and ω are polynomials in y, the operator R can be polynomial in y. In this case, applying theorem 7, we obtain an operator R satisfying the conditions (A) and (B). Then the condition (C) is checked.

Example. Consider the control system

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= \sin\frac{u_1}{u_2}. \end{aligned} \tag{46}$$

To find a B–basis, note that any element of $\mathcal{H}_0$ has the form

$$\omega = \sum_{i=1}^{3} a_i d_{\mathcal{C}} x_i, \qquad a_i \in \mathcal{F}(\mathcal{E}).$$

We have

$$D\omega = \sum_{i=1}^{3} Da_i d_{\mathcal{C}} x_i + a_1 d_{\mathcal{C}} u_1 + a_2 d_{\mathcal{C}} u_2 + a_3 \cos(u_1/u_2)\left(\frac{1}{u_2} d_{\mathcal{C}} u_1 - \frac{u_1}{u_2^2} d_{\mathcal{C}} u_2\right). \tag{47}$$

If $D\omega \in \mathcal{H}_0$, then the coefficients at $d_{\mathcal{C}} u_1$ and $d_{\mathcal{C}} u_2$ in (46) vanish. Whence,

$$a_1 + a_3 \cos(u_1/u_2)\frac{1}{u_2} = 0, \qquad a_2 - a_3 \cos(u_1/u_2)\frac{u_1}{u_2^2} = 0.$$

Therefore the module $\mathcal{H}_1$ is generated by

$$\omega_1 = d_{\mathcal{C}} x_1 - \frac{u_1}{u_2} d_{\mathcal{C}} x_2 - \frac{u_2}{\cos(u_1/u_2)} d_{\mathcal{C}} x_3.$$

In the same way, the condition $D(f\omega_1) \in \mathcal{H}_1$ means that $f = 0$. We see that $k^* = 2$ and $\rho = 0$. The 1-form ω_2 should be chosen such that $\{\omega_1, D\omega_1, \omega_2\}$ is a basis of the module $\mathcal{H}_0$. We put $\omega_2 = d_{\mathcal{C}} x_3$. In this case, the set of B-regular points is $\{D(u_1/u_2) \neq 0\}$ and $\omega = \{\omega_1, \omega_2\}$ is a B–basis.

To obtain a polynomial form for D and ω, we introduce the following control variables:

$$v_1 = \frac{u_1}{u_2}, \qquad v_2 = u_2.$$

It is readily seen that D and ω are polynomials in the variables $x_2, v_2, D^i v_j, i > 0, j = 1, 2$. The corresponding coordinates on the diffiety are $x_1, x_2, x_3, v_1, v_2, D^i v_j, i > 0, j = 1, 2$. We have

$$\omega_{(0)} = \begin{pmatrix} d_{\mathcal{C}} x_1 \\ d_{\mathcal{C}} x_3 \end{pmatrix}, \qquad \omega_{(1)} = \begin{pmatrix} -v_1 d_{\mathcal{C}} x_2 \\ 0 \end{pmatrix}.$$

Since $d_{\mathcal{C}} \omega_{(0)} = 0$, we can put $R_{(0)} = 0$. Equations (34) for $j = 0$ have the form

$$R^0_{(1)}(\omega_{(0)}) = -d_{\mathcal{C}} \omega_{(1)}, \qquad [d_{\mathcal{C}}, R^0_{(1)}]_+ = 0.$$

Whence,

$$R_{(1)} = \begin{pmatrix} 0 & -d_{\mathcal{C}}(x_2 \cos v_1) \wedge D \\ 0 & 0 \end{pmatrix}.$$

It is clear that the operator $R = R_{(1)}$ satisfies the conditions (A), (B), and (C).

To find the operator Δ, note that the operators from (C) have the form

$$\begin{pmatrix} 0 & gD \\ 0 & 0 \end{pmatrix}, \quad g \in \mathcal{F}(\mathcal{E}).$$

Consequently the module C_R consists of all the operators of the form

$$\Delta = \begin{pmatrix} f_1 & f_3 D + f_5 \\ f_2 & f_4 D + f_6 \end{pmatrix}, \quad f_i \in \mathcal{F}(\mathcal{E}).$$

Therefore equation (13) has the form

$$\begin{pmatrix} d_{\mathcal{C}} f_1 & d_{\mathcal{C}} f_3 \wedge D + d_{\mathcal{C}} f_5 \\ d_{\mathcal{C}} f_2 & d_{\mathcal{C}} f_4 \wedge D + d_{\mathcal{C}} f_6 \end{pmatrix} = \begin{pmatrix} 0 & -f_1 d_{\mathcal{C}}(x_2 \cos v_1) \wedge D \\ 0 & -f_2 d_{\mathcal{C}}(x_2 \cos v_1) \wedge D \end{pmatrix}.$$

So we get

$$\begin{aligned} &d_{\mathcal{C}} f_3 = -f_1 d_{\mathcal{C}}(x_2 \cos v_1), \qquad && d_{\mathcal{C}} f_4 = -f_2 d_{\mathcal{C}}(x_2 \cos v_1), \\ & && d_{\mathcal{C}} f_1 = d_{\mathcal{C}} f_2 = d_{\mathcal{C}} f_5 = d_{\mathcal{C}} f_6 = 0. \end{aligned} \tag{48}$$

Whence f_i are functions of t, x_2, and v_1. Let s be a solution $(x_i = q_i(t), v_j = p_j(t))$ of (45). From the initial condition $\Delta|_s = \mathrm{id}$ and (47) it follows that

$$f_1 = 1, \quad f_2 = 0, \quad f_3 = -x_2 \cos v_1 + q_2(t) \cos p_1(t), \quad f_4 = f_5 = 0, \quad f_6 = 1.$$

Thus

$$\Delta = \begin{pmatrix} 1 & \Big(-x_2 \cos v_1 + q_2(t) \cos p_1(t)\Big) D \\ 0 & 1 \end{pmatrix}.$$

It is easy to prove that for any functions $q_2(t)$ and $p_1(t)$ the operator Δ satisfies condition (13). Take $q_2(t) = p_1(t) = 0$. From Theorem 2, we obtain flat outputs $h_1 = x_1 - x_2 u_1 / u_2, h_2 = x_3$.

References

[1] Aranda–Bricaire, E., Moog, C. H., Pomet J.-B.: An infinitesimal Brunovsky form for nonlinear systems with applications to dynamic linearization. In: Geometry in Nonlinear Control and Differential Inclusions (ed. by B. Jakubczyk, W. Respondek, and T. Rzezuchowski), pp. 19-33. Banach Center Publications, 32, Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland 1995

[2] Chetverikov, V. N.: Higher symmetries and infinitesimal Brunovský form of control systems. Nauchniy Vestnik MGTU GA 16 (1999), 77-83 (in Russian).

[3] Fliess, M., Lévine, J., Martin, Ph., Rouchon, P.: Flatness and defect of non-linear systems: Introductory theory and examples. Int. J. Contr. 61, n. 6 (1995), 1327-1361.

[4] Fliess, M., Lévine, J., Martin, Ph., Rouchon, P.: A Lie–Bäcklund approach to equivalence and flatness of nonlinear systems. IEEE Trans. Automat. Control 44, n. 5 (1999), 922-937.

[5] Guillemin, V., Sternberg, S.: Deformation theory of pseudogroup structures. Mem. Amer. Math. Soc. 64 (1966).

[6] Krasil'shchik, I. S., Lychagin, V. V., Vinogradov, A. M.: Geometry of Jet Spaces and Nonlinear Partial Differential Equations. Gordon and Breach, New York 1986

[7] Krasil'shchik, I. S., Vinogradov, A. M., Eds.: Symmetries and Conservation Laws for Differential Equations of Mathematical Physics. Amer. Math. Soc., Providence, RI 1999

[8] Pomet, J.-B.: A differential geometric setting for dynamic equivalence and dynamic linearization. In: Geometry in Nonlinear Control and Differential Inclusions (ed. by B. Jakubczyk, W. Respondek, and T. Rzezuchowski), pp. 319-339. Banach Center Publications, 32, Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland 1995

[9] Spencer, D. C.: Deformation of structures on manifolds defined by transitive continuous pseudogroups. I: Annals of Math. 76 (1962), p. 306-445, II, III: Annals of Math. 81 (1965), p. 389-450.

Bauman Moscow State Technical University, ul. 2-aja Baumanskaja, 5, 107005, Russia, Moscow

E-mail address: chetveri@diffiety.ac.ru