The Diffiety Institute Preprint Series

Preprint DIPS–4/2006 **October 30, 2006**

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The Diffiety Institute

DIPS-4/2006
arXiv:math.DG/0610914

Iterated Differential Forms III: Integral Calculus

A. M. VINOGRADOV AND L. VITAGLIANO

ABSTRACT. Basic elements of integral calculus over algebras of iterated differential forms Λ_k, $k < \infty$, are presented. In particular, defining complexes for modules of integral forms are described and the corresponding berezinians and complexes of integral forms are computed. Various applications and the integral calculus over the algebra Λ_∞ will be discussed in subsequent notes.

1. Integral Calculus over Graded Algebras

In this note we follow the notation and definitions of [1, 2] (see also [3, 4, 5, 6]). The necessary generalities concerning integral calculus over graded commutative algebras are collected below by following the approach of [7, 8].

Let $\mathcal{G} = (G, \mu)$ be a grading group, $\Bbbk$ a field of zero–characteristic and A a $\mathcal{G}$–graded commutative algebra. $\Lambda(A) = \bigoplus_s \Lambda^s(A)$ stands for the $\Bbbk$–algebra of differential forms over A, which is naturally $\mathcal{G} \oplus \mathbb{Z}$–graded. So, the de Rham complex over A reads

$$0 \longrightarrow A \xrightarrow{d} \Lambda^1(A) \xrightarrow{d} \cdots \longrightarrow \Lambda^s(A) \xrightarrow{d} \cdots$$

The A-module of graded, skew–symmetric, s–derivations of A with values in an A–module P is denoted by $\mathrm{D}_s(A, P)$. Put $\mathrm{D}_\bullet(A, P) = \bigoplus_{s \geq 0} \mathrm{D}_s(A, P)$ assuming that $\mathrm{D}_0(A, P) = P$. A–modules $\mathrm{Diff}_A(P, \Lambda^s(A))$, $s \geq 0$, of graded linear differential operators that send P to $\Lambda^s(A)$ form the complex

$$0 \longrightarrow \mathrm{Diff}_A(P, A) \xrightarrow{w_P} \mathrm{Diff}_A(P, \Lambda^1(A)) \xrightarrow{w_P} \cdots \longrightarrow \mathrm{Diff}_A(P, \Lambda^s(A)) \xrightarrow{w_P} \cdots \quad (1)$$

with $w_P(\Delta) = d \circ \Delta$, $\Delta \in \mathrm{Diff}_A(P, \Lambda(A))$. The cohomology $\widehat{P} \stackrel{\text{def}}{=} H(w_P)$ of complex (1) carries a natural graded A–module structure given by

$$a[\Delta] = (-1)^{|a| \cdot |\Delta|}[\Delta \circ a], \quad a \in A,\ [\Delta] \in \widehat{P},$$

$\Delta \in \mathrm{Diff}_A(P, \Lambda(A))$, $w_P(\Delta) = 0$. A–module $\widehat{P}$ is called *adjoint to* P and (1) the *defining complex of* $\widehat{P}$.

2000 *Mathematics Subject Classification.* 58A10, 58A50, 13N99.
Key words and phrases. Differential Calculus, Graded Algebras, Differential Forms, Integral Forms.

A graded linear differential operator $\square : P \longrightarrow Q$, Q being an A–module, induces the co-chain map

$$\widetilde{\square} : \mathrm{Diff}_A(Q, \Lambda(A)) \ni \Delta \longmapsto (-1)^{|\Delta|\cdot|\square|}\Delta \circ \square \in \mathrm{Diff}_A(P, \Lambda(A)),$$

i.e., $w_Q \circ \widetilde{\square} = \widetilde{\square} \circ w_P$. The corresponding map in cohomology $\widehat{\square} : \widehat{Q} \longrightarrow \widehat{P}$, which is a differential operator, is called the *adjoint to* $\square$ *operator*.

Elements of A–module $\Sigma_s(A) \stackrel{\text{def}}{=} \widehat{\Lambda^s(A)}$ are called *integral* s*–forms* over A and $\mathcal{B}(A) \stackrel{\text{def}}{=} \widehat{A} = \Sigma_0(A)$ is called the *berezinian* of A. The module $\Sigma(A) = \bigoplus_{s\geq 0} \Sigma_s(A)$ is naturally supplied with a $\mathcal{G} \oplus \mathbb{Z}$–graded right $\Lambda(A)$–module structure given by

$$\Sigma_{l+m}(A) \times \Lambda^l(A) \ni (Z, \omega) \longmapsto \langle Z, \omega\rangle \stackrel{\text{def}}{=} [\nabla \circ \omega] \in \Sigma_m(A) \tag{2}$$

where $Z = [\nabla] \in \Sigma_{l+m}(A)$, $\nabla \in \mathrm{Diff}_A(\Lambda^{l+m}(A), \Lambda(A))$, $d \circ \nabla = 0$.

The complex of integral forms is the adjoint to the de Rham one:

$$0 \longleftarrow \mathcal{B}(A) \stackrel{\widehat{d}}{\longleftarrow} \Sigma_1(A) \longleftarrow \cdots \stackrel{\widehat{d}}{\longleftarrow} \Sigma_s(A) \longleftarrow \cdots$$

The following "*right*" *Leibnitz rule* holds

$$\langle \widehat{d}Z, \omega\rangle = \langle Z, d\omega\rangle + (-1)^l \widehat{d}\,\langle Z, \omega\rangle\,, \tag{3}$$

with $Z \in \Sigma(A)$, $\omega \in \Lambda^l(A)$.

For an A–module P denote by $\mathrm{Diff}^{>}_A(P, \Lambda(A))$ the right A–module structure on the vector space of linear differential operators acting on P and with values in $\Lambda(A)$, i.e.,

$$a^{>}\square \stackrel{\text{def}}{=} (-1)^{|a|\cdot|\square|}\square \circ a, \quad a \in A,\ \square \in \mathrm{Diff}^{>}_A(P, \Lambda(A)).$$

There takes place a natural isomorphism (see, e.g., [7])

$$\mathrm{Diff}^{>}_A(\Lambda^s(A), \Lambda(A)) \ni \square \longmapsto X_\square \in \mathrm{D}_s(A, \mathrm{Diff}^{>}_A(A, \Lambda(A)))$$

and a natural pairing

$$\mathrm{D}_{l+m}(A, \mathcal{B}(A)) \times \Lambda^l(A) \ni (X, \omega) \longmapsto \langle\!\langle X, \omega\rangle\!\rangle \in \mathrm{D}_m(A, \mathcal{B}(A)) \tag{4}$$

defined by

$$\langle\!\langle X, \omega\rangle\!\rangle(a_1, \ldots, a_m) \stackrel{\text{def}}{=} i_X(\omega \wedge da_1 \wedge \cdots \wedge da_m).$$

$a_1, \ldots, a_m \in A$. The pairing (4) supplies $\mathrm{D}_\bullet(A, \mathcal{B}(A))$ with a graded right $\Lambda(A)$–module structure.

A natural 0–degree homomorphism of right $\Lambda(A)$–modules

$$\chi_A : \Sigma(A) \longrightarrow \mathrm{D}_\bullet(A, \mathcal{B}(A)) \tag{5}$$

is defined as follows. If $Z = [\nabla] \in \Sigma_s(A)$, $\nabla \in \mathrm{Diff}_A(\Lambda^s(A), \Lambda(A))$, then

$$\chi_A(Z)(a_1, \ldots, a_s) \stackrel{\text{def}}{=} [X_\nabla(a_1, \ldots, a_s)] \in \mathcal{B}(A), \quad a_1, \ldots, a_s \in A.$$

Namely,

$$\chi_A(\langle Z, \omega\rangle) = \langle\!\langle \chi_A(X), \omega\rangle\!\rangle, \quad Z \in \Sigma(A),\ \omega \in \Lambda(A).$$

Proposition 1. *If $\Lambda^1(A)$ is a projective and finitely generated A–module, then χ_A is an isomorphism.*

In other words, in a smooth situation the isomorphism χ_A gives an exact description of integral forms exclusively in terms of the berezinian $\mathcal{B}(A)$.

2. Defining Complex of Integral Forms over Iterated Differential Form Algebras

In this section the complex of integral forms over the algebra of geometric iterated differential forms on a smooth manifold is described. This description and the results presented in the rest of this note are generalized almost automatically to any "smooth" situation, for instance, to super-manifolds. However, for simplicity's sake we do not report them here. In what follows M stands for a smooth n–dimensional manifold, $(x^1, \ldots, x^n)$ for a local chart in it and all functors of differential calculus over commutative algebras and representing them objects are specialized to the category of geometric modules over the algebra $C^\infty(M)$ in order to be in conformity with the standard differential geometry (see [6]). Accordingly, below $\Lambda_k = \Lambda_k(M)$ stands for k times iterated geometric differential forms over $C^\infty(M)$ (see [1]). Put also $\Lambda^s_{k+1} = \Lambda^s(\Lambda_k) \subset \Lambda_{k+1}$ and note that Λ^s_{k+1} is a projective Λ_k–module and Λ^1_{k+1} is locally freely generated by elements $d_{k+1}d_L x^\mu$, $\mu = 1, \ldots, n$, $L \subset \{1, \ldots, k\}$. Elements of the dual basis are denoted by $\frac{\partial}{\partial d_L x^\mu} \in \mathrm{D}(\Lambda_k, \Lambda_k) \simeq \mathrm{Hom}_{\Lambda_k}(\Lambda^1_{k+1}, \Lambda_k)$, $\mu = 1, \ldots, n$, $L \subset \{1, \ldots, k\}$, i.e.,

$$\frac{\partial}{\partial d_L x^\mu}(d_J x^\nu) = \begin{cases} 1, & \text{if } \nu = \mu \text{ and } J = L \\ 0, & \text{otherwise} \end{cases} .$$

Integral geometric p–forms over Λ_k are cohomology classes of the complex

$$0 \longrightarrow \mathrm{Diff}_{\Lambda_k}(\Lambda^p_{k+1}, \Lambda_k) \xrightarrow{w_{k,p}} \mathrm{Diff}_{\Lambda_k}(\Lambda^p_{k+1}, \Lambda^1_{k+1}) \xrightarrow{w_{k,p}} \cdots \longrightarrow \mathrm{Diff}_{\Lambda_k}(\Lambda^p_{k+1}, \Lambda^s_{k+1}) \xrightarrow{w_{k,p}} \cdots$$

where $w_{k,p}(\Delta) = d_{k+1} \circ \Delta$, $\Delta \in \mathrm{Diff}_{\Lambda_k}(\Lambda^p_{k+1}, \Lambda_{k+1})$. The Λ_k–module $\mathrm{Diff}_{\Lambda_k}(\Lambda_k, \Lambda^s_{k+1})$ is locally freely generated by elements

$$d_{k+1}d_{L_1}x^{\mu_1} \wedge \cdots \wedge d_{k+1}d_{L_s}x^{\mu_s} \wedge \frac{\partial}{\partial d_{J_1}x^{\nu_1}} \circ \cdots \circ \frac{\partial}{\partial d_{J_r}x^{\nu_r}}, \quad r \geq 0$$

with $\mu_1, \ldots, \mu_s, \nu_1, \ldots, \nu_r = 1, \ldots, n$ and $L_1, \ldots, L_s, J_1, \ldots, J_r \subset \{1, \ldots, k\}$.

3. Berezinian of the Algebra of Iterated Differential Forms and Adjoint de Rham Differentials

Put $\nu(k) \stackrel{\text{def}}{=} 2^{k-1}n$.

Theorem 2.

(i) $H^s(w_{k,0}) = 0$, *if* $s \neq \nu(k)$,

(ii) *There exists a natural isomorphism of Λ_k–modules* $\beta_k : \Lambda_k \longrightarrow \mathcal{B}(\Lambda_k) = H^{\nu(k)}(w_{k,0})$.

It is worth stressing that the definition of the Berezinian we use in this note is different from the standard one in the theory of super-manifolds (see, for instance, [9, 10]). Even though these definitions are, as it can be shown, equivalent (see [8]) the *conceptuality* of the former makes it much more preferable for our goals.

According to theorem 2 the element $\zeta_k = \beta_k(1_{\Lambda_k})$ freely generates Λ_k–module $\mathcal{B}(\Lambda_k)$ and its local description is as follows. Let $\mathcal{U} = \{(x^1, \ldots, x^n)\}$ be a local chart and $I_1, \ldots, I_{2^{k-1}}$ (resp., $J_1, \ldots, J_{2^{k-1}}$) be all subsets of $\{1, \ldots, k\}$ composed of an even (resp., odd) number of elements. It is assumed that each of these two families of subsets is ordered once for all according to the subscripts. Put

$$\Omega_{\mathcal{U}} \overset{\text{def}}{=} d_{k+1}d_{I_1}x^1 \wedge \cdots \wedge d_{k+1}d_{I_{2^{k-1}}}x^1 \wedge \cdots \wedge d_{k+1}d_{I_1}x^n \wedge \cdots \wedge d_{k+1}d_{I_{2^{k-1}}}x^n \in \Lambda_{k+1}^{\nu(k)}(\mathcal{U}),$$

$$\Delta_{\mathcal{U}} \overset{\text{def}}{=} \tfrac{\partial}{\partial d_{J_1}x^1} \circ \cdots \circ \tfrac{\partial}{\partial d_{J_{2^{k-1}}}x^1} \circ \cdots \circ \tfrac{\partial}{\partial d_{J_1}x^n} \circ \cdots \circ \tfrac{\partial}{\partial d_{J_{2^{k-1}}}x^n} \in \mathrm{Diff}_{\Lambda_k(\mathcal{U})}(\Lambda_k(\mathcal{U}), \Lambda_k(\mathcal{U})).$$

Then $\zeta_k|_{\mathcal{U}} = [\square] \in \mathcal{B}(\Lambda_k(\mathcal{U}))$ with

$$\square \overset{\text{def}}{=} \Omega_{\mathcal{U}} \wedge \Delta_{\mathcal{U}} \in \mathrm{Diff}_{\Lambda_k(\mathcal{U})}(\Lambda_k(\mathcal{U}), \Lambda_{k+1}^{\nu(k)}(\mathcal{U})).$$

Now define the homomorphism $\chi_{k,s} : \Sigma_s(\Lambda_k) \longrightarrow \mathrm{D}_s(\Lambda_k, \Lambda_k)$ by putting

$$(\chi_{k,s}(Z)(\omega_1, \ldots, \omega_s))\, \zeta_k = \chi_{\Lambda_k}(Z)(\omega_1, \ldots, \omega_s), \quad Z \in \Sigma_s(\Lambda_k),$$

(see section 1).

Theorem 3. *$\chi_{k,s}$ is an isomorphism of Λ_k–modules.*

So, $\Sigma_s(\Lambda_k)$ is identified with $\mathrm{D}_s(\Lambda_k, \Lambda_k)$ via $\chi_{k,s}$ and, in view of this identification, the complex of integral forms over Λ_k reads

$$\Lambda_k \overset{\widehat{d}_{k+1}}{\longleftarrow} \mathrm{D}(\Lambda_k, \Lambda_k) \longleftarrow \cdots \overset{\widehat{d}_{k+1}}{\longleftarrow} \mathrm{D}_s(\Lambda_k, \Lambda_k) \longleftarrow \cdots .$$

A description of the differential $\widehat{d}_{k+1}$ in these terms is as follows.

Let $Z \in \mathrm{D}_s(\Lambda_k, \Lambda_k)$. Note that the multi-derivation $\widehat{d}_{k+1}Z \in \mathrm{D}_{s-1}(\Lambda_k, \Lambda_k)$ is completely determined by its values $\langle \widehat{d}_{k+1}Z, \Omega\rangle$ on forms $\Omega \in \Lambda_{k+1}^{s-1}$. But in view of (3) computation of these values is reduced to computation of the action of $\widehat{d}_{k+1}$ on integral 1–forms, i.e., on $\mathrm{D}(\Lambda_k, \Lambda_k)$. The latter is described in the following proposition.

Proposition 4. *If $Z \in \Sigma_1(\Lambda_k) \simeq \mathrm{D}(\Lambda_k, \Lambda_k)$, then $\widehat{d}_{k+1}Z = \widehat{Z}(\zeta_k)$ with $\widehat{Z} : \mathcal{B}(\Lambda_k) \simeq \Lambda_k \longrightarrow \mathcal{B}(\Lambda_k) \simeq \Lambda_k$ being the adjoint to Z operator.*

If, locally,

$$Z = \sum_{\mu, L} Z_L^\mu \tfrac{\partial}{\partial d_L x^\mu}, \quad Z_L^\mu \in \Lambda_k,$$

with the summation running over $\mu = 1, \ldots, n$ and all subsets L of $\{1, \ldots, k\}$, then

$$\widehat{d}_{k+1}Z = -\sum_{\mu, L} (-1)^{|d_L|\cdot(|d_L| + |Z|)} \tfrac{\partial}{\partial d_L x^\mu}(Z_L^\mu).$$

Theorem 5. $H_i(\widehat{d_{k+1}}) \simeq H^{\nu(k)-i}(\Lambda_1, d_1)$.

As an example we give an explicit description of the adjoint operator $\widehat{d_l}$: $\mathrm{D}_s(\Lambda_k, \Lambda_k) \longrightarrow \mathrm{D}_s(\Lambda_k, \Lambda_k)$ to the l–th iterated de Rham differential $d_l : \Lambda^s_{k+1} \longrightarrow \Lambda^s_{k+1}$ for $l \leq k$.

Proposition 6. *Let* $Z \in \mathrm{D}_s(\Lambda_k, \Lambda_k)$. *Then*

$$\begin{aligned}(\widehat{d_l}Z)(\omega_1, \dots, \omega_s) = &\textstyle\sum_l (-1)^{|d_l|(|Z|+|\omega_1|+\cdots+|\omega_{i-1}|)} Z(\omega_1, \dots, \omega_{i-1}, d_l\omega_i, \omega_{i+1}, \dots, \omega_s) \\ &- d_l(Z(\omega_1, \dots, \omega_s)).\end{aligned}$$

4. The Trace of an Endomorphism

Now we shall illustrate the developed theory by a simple example showing that the notion of trace is a natural part of integral calculus. Consider with this purpose an endomorphism of the tangent bundle of M, or, equivalently, a (1,1)-tensor field on M. It is naturally interpreted as a $\Lambda^1(M)$–valued derivation of the algebra $C^\infty(M)$ and hence as an integral form over the algebra $\Lambda = \Lambda(M)$. Namely, let $X \in \mathrm{D}(C^\infty(M), \Lambda^1(M))$. According to the standard definition, $\operatorname{tr} X$, the trace of X, is the smooth function on M uniquely determined by the identity $i_X(\omega) = \operatorname{tr} X\, \omega$, $\omega \in \Lambda^n(M)$, that can be rewritten as

$$p_n \circ i_X = \operatorname{tr} X\, p_n,$$

where $p_n : \Lambda(M) \longrightarrow \Lambda^n(M)$ is a natural projection.

Now, according to the previous section $\zeta_1 = [\Box] \in \mathcal{B}(\Lambda)$, where $\Box \in \mathrm{Diff}_\Lambda(\Lambda, \Lambda_2)$ is locally given by

$$\Box = d_2x^1 \wedge \cdots \wedge d_2x^n \wedge \tfrac{\partial}{\partial d_1 x^1} \circ \cdots \circ \tfrac{\partial}{\partial d_1 x^n}.$$

The coordinate-free description of $\Box$ is

$$\Box = (-1)^{n(n-1)/2} \kappa \circ p_n,$$

with $\kappa : \Lambda_2 \longrightarrow \Lambda_2$ being the involution that interchanges differentials d_1 and d_2 (see [1]). Now, remembering that $i_X \in \mathrm{D}(\Lambda, \Lambda)$ is an integral 1–form over Λ, we have:

$$\widehat{d_2}(i_X) \simeq \widehat{i_X}(\zeta_1) = [\Box \circ i_X] = (-1)^{n(n-1)/2}[\kappa \circ p_n \circ i_X] = \operatorname{tr} X\, \zeta_1.$$

Since ζ_1 is a *canonical* integral form, this result allows to identify $\operatorname{tr} X$ with the integral form $\widehat{d_2}(i_X) \in \mathcal{B}(\Lambda) = \Lambda$.

5. Final Remarks

Our interest in integral calculus with iterated differential forms is motivated by the problem of unification of various natural integration procedures that one meets in differential geometry, theoretical physics, etc. From what was said above one can see that the developed calculus puts in common frames usual integrals, interpreted as the de Rham cohomology, and traces of endomorphisms. These facts give some simple arguments in favor of iterated forms in this connection. Further results, which this approach allows to get on in this direction, will be given in separate publications.

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