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by

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ABSTRACT. In this paper a description of the term E_1 of the Vinogradov $\mathcal{C}$-spectral sequence is given. Vanishing theorem for the group $E_1^{p,n-1}$, where n is a number of independent variables and $p \geq 3$, is proved for a system of evolution equations with a non-degenerate symbol. The group $E_1^{2,n-1}$ for a scalar linear evolution equation with constant coefficients is computed.

1. INTRODUCTION

Homological methods play an important role in the study of systems of differential equations. In [4] it is shown that infinitesimal symmetries of differential equations and recursion operators are of cohomological nature. The $\mathcal{C}$-spectral sequence (variational bicomplex) introduced by Vinogradov [8] contains important invariants of differential equations such as conservation laws and characteristic classes of families of solutions. It provides means for studying different aspects of Lagrangian formalism, the inverse problem of the calculus of variations, etc. The term E_1 of the $\mathcal{C}$-spectral sequence is the analog of the de Rham complex in the category of nonlinear partial differential equations (for a very enlightening discussion see [12]).

There exists, however, no general method of computing this important spectral sequence. In [10] a powerful technique based on the Spencer type cohomology systems is developed, the "two line" theorem estimating the term E_1 for determined systems of differential equations is proved, a concrete method of calculating for the term $E_1^{1,n-1}$, which is related to the theory of conservation laws, is given, where n is a number of independent variables. Due to this method a complete description of the set of conservation laws is possible for determined differential equations (see, for example, [11]). Further development of results of Vinogradov is done in [7, 6], where the Janet sequence for involutive

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differential equations is used, and a general approach to the calculation of the horizontal cohomology is proposed. In [2] this approach is applied to the $\mathcal{C}$-spectral sequence for overdetermined equations.

Some results concerning the computation of the term $E_1^{p,n-1}$, $p \geq 2$, are obtained in [3], where the $\mathcal{C}$-spectral sequence is considered for scalar evolution equations with a single space variable. For a large class of such equations, in particular KdV, MKdV, PKdV, Burgers equation, the term $E_1^{2,1}$ is computed and vanishing theorems are proved for the groups $E_1^{p,n-1}$, $p \geq 3$, under a very restrictive assumption that an equation possesses an infinite dimensional space of higher infinitesimal symmetries. Some examples of computation of the $\mathcal{C}$-spectral sequence also can be found in [1].

In this paper we develop a method of calculating the terms $E_1^{p,n-1}$, $p \geq 1$, for determined differential equations and apply this method to evolution equations. This paper is organized as follows. Section 2 is a summary of the geometrical theory of nonlinear differential equations and the $\mathcal{C}$-spectral sequence. In Section 3 we give a description of the term E_1 of the $\mathcal{C}$-spectral sequence for determined equations. In Section 4 this general description is applied to evolution equations, vanishing theorem for the group $E_1^{p,n-1}$, $p \geq 3$, is proved for evolution equation with a nondegenerate symbol, the group $E_1^{2,n-1}$ is computed for a scalar linear evolution equation with constant coefficients.

2. Jet manifolds and infinitely prolonged differential equations

In this section we define the basic concepts of the geometrical theory of differential equations and the theory of the $\mathcal{C}$-spectral sequence ([5, 10, 9, 1]).

2.1. **Jets.** Let M be a smooth manifold and $\pi : E \to M$ be a smooth fiber bundle over M, $\dim M = n$, $\dim E = m + n$. Denote by $\Gamma(\pi)$ the set of all (local) sections of π.

Let $\pi_k : J^k \to M$ be *the bundle of k-jets* for π,

$$J^k(\pi) = \left\{ [f]_x^k \mid f \in \Gamma(\pi), x \in M \right\},$$

where $[f]_x^k$ denotes the k-jet of a local section f at x. Denote by $J^\infty(\pi)$ the manifold of infinite jets for π. $J^\infty(\pi)$ is the inverse limit with respect to the following system of mappings

$$\pi_{k,l} : J^k(\pi) \to J^l(\pi), \quad \pi_{k,l}\left([f]_x^k\right) = [f]_x^l, k \geq l,$$
$$\pi_k : J^k(\pi) \to M, \pi_k\left([f]_x^k\right) = x.$$

By definition, one has natural projections

$$\pi_{\infty,k} : J^\infty(\pi) \to J^k(\pi),$$
$$\pi_\infty : J^\infty(\pi) \to M.$$

Choose a coordinate neighborhood U in M such that the restriction of the bundle E is trivial. Let $x_1, \dots, x_n$ be local coordinates in U and $u^1, \dots, u^m$ be coordinates along the fiber of π in $\pi^{-1}(U)$. Each local section $f \in \Gamma(\pi)$ is of the form $f = (u^1(x_1, \dots, x_n), \dots, u^m(x_1, \dots, x_n))$. Define functions p^j_σ by following

$$p^j_\sigma\left([f]^k_{x_0}\right) = \left.\frac{\partial^{|\sigma|} u^j}{\partial x_1^{\sigma_1} \dots \partial x_n^{\sigma_n}}\right|_{x=x_0},$$

where $\sigma = (\sigma_1, \dots, \sigma_n), |\sigma| = \sigma_1 + \dots + \sigma_n$. Then smooth functions (x^i, p^j_σ), $1 \le i \le n$, $1 \le j \le m$, $0 \le |\sigma| \le k$ form local coordinates in $J^k(\pi)$, $0 \le k \le \infty$.

Let $\mathcal{F}_k(\pi)$ denote the algebra $C^\infty\left(J^k(\pi)\right)$. Then one has a system of embeddings

$$\pi^*_{k,l} : \mathcal{F}_l(\pi) \to \mathcal{F}_k(\pi), l \le k,$$
$$\pi^*_k : C^\infty(M) \to \mathcal{F}_k(\pi).$$

The direct limit $\mathcal{F}(\pi)$ with respect to the system $\{\pi^*_{k,l}\}$ is called *the algebra of smooth functions* on $J^\infty(\pi)$. We identify $\mathcal{F}_k(\pi)$ and $C^\infty(M)$ with their limits in $\mathcal{F}(\pi)$,

$$\mathcal{F}(\pi) = \bigcup_{k \ge 0} \mathcal{F}_k(\pi).$$

In the same manner one can define the module of *i-forms* $\Lambda^i(\pi)$, $i \ge 0$, on $J^\infty(\pi)$

$$\Lambda^i(\pi) = \bigcup_{k \ge 0} \Lambda^i\left(J^k(\pi)\right)$$

and consider a graded algebra

$$\Lambda^*(\pi) = \sum_{i=0}^{\infty} \Lambda^i(\pi)$$

of differential forms on $J^\infty(\pi)$.

A *vector field* X on $J^\infty(\pi)$ is an $\mathbb{R}$-linear mapping $X : \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ such that for any $\varphi, \psi \in \mathcal{F}(\pi)$

$$X(\varphi\psi) = \varphi X(\psi) + \psi X(\varphi),$$

and there exists $r \ge 0$ such that for each $k \ge 0$

$$X\left(\mathcal{F}_k(\pi)\right) \subset \mathcal{F}_{k+r}(\pi).$$

Denote by $D(\pi)$ the set of all vector fields on $J^\infty(\pi)$. Obviously, $D(\pi)$ is a $\mathcal{F}(\pi)$-module and Lie algebra over $\mathbb{R}$. Locally each $X \in D(\pi)$ can be represented as

$$X = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_{j,\sigma} b^j_\sigma \frac{\partial}{\partial p^j_\sigma},$$

where $b^j_\sigma \in \mathcal{F}_{|\sigma|+r}(\pi)$ for some r.

Let $\xi_i : F_i \to J^\infty(\pi)$, $i = 1, 2$, be vector bundles over $J^\infty(\pi)$, $P_i = \Gamma(F_i)$ be $\mathcal{F}(\pi)$-modules of sections. An $\mathbb{R}$-linear differential operator $\Delta : P_1 \to P_2$ is called *$\mathcal{C}$-differential* if it can be restricted to the manifolds of the form $[f]^\infty$, $f \in \Gamma(\pi)$. That is

$$\Delta(\varphi)|_{[f]^\infty} = 0 \text{ as } \varphi|_{[f]^\infty} = 0, \ \varphi \in P_1.$$

If vector bundles ξ_1, ξ_2 are finite dimensional, $m_i = \dim \xi_i$, then in local coordinates each $\mathcal{C}$-differential operator Δ can be represented as a $m_2 \times m_1$ matrix

$$\Delta = \begin{pmatrix} \sum_\sigma a^\sigma_{11} D_\sigma & \dots & \sum_\sigma a^\sigma_{1m_1} D_\sigma \\ \dots & \dots & \dots \\ \sum_\sigma a^\sigma_{m_2 1} D_\sigma & \dots & \sum_\sigma a^\sigma_{m_2 m_1} D_\sigma \end{pmatrix},$$

where $\sigma = (\sigma_1, \dots, \sigma_n)$, $D_\sigma = (D_1)^{\sigma_1} \circ \dots \circ (D_n)^{\sigma_n}$, and

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} p^j_{\sigma+1_i} \frac{\partial}{\partial p^j_\sigma}, i = 1, \dots, n,$$

is the *i-th total derivative*.

The set of $\mathcal{C}$-differential operators from P_1 to P_2 is, clearly, a $\mathcal{F}(\pi)$-module and is denoted by $\mathcal{C}\mathrm{Diff}(P_1; P_2)$.

Let $\xi_i : F_i \to M$, $i = 1, 2$, be vector bundles, $P_i = \Gamma(\xi_i)$, and $\Delta : P_1 \to P_2$ be an $\mathbb{R}$-linear differential operator, $\Delta \in \mathrm{Diff}(P_1; P_2)$. By definition, put $\overline{P_i} = \mathcal{F}(\pi) \otimes_{C^\infty(M)} P_i = \Gamma(\pi^*_\infty(\xi_i))$, where $\pi^*_\infty(\xi_i)$ is the induced vector bundle over $J^\infty(\pi)$. A unique $\mathcal{C}$-differential operator $\overline{\Delta} : \overline{P_1} \to \overline{P_2}$ such that $\overline{\Delta}(1 \otimes p) = 1 \otimes \Delta(p)$ is called *the lifting of* Δ.

Consider a vector field $X \in D(M)$ on M. It is an operator of the first order acting from $C^\infty(M)$ to $C^\infty(M)$. Then the lifting $\overline{X} \in \mathcal{C}\mathrm{Diff}\,(\mathcal{F}(\pi), \mathcal{F}(\pi))$ is a vector field on $J^\infty(\pi)$. Consider a submodule $\mathcal{C}D(\pi) \subset D(\pi)$ generated by vector fields of the form $\overline{X}$. Thus we have a distribution on $J^\infty(\pi)$ which is called the *Cartan distribution*. The Cartan distribution is completely integrable in the sense that it satisfies Frobenius integrability condition

$$[\mathcal{C}D(\pi), \mathcal{C}D(\pi)] \subset \mathcal{C}D(\pi). \tag{1}$$

In local coordinates if $X = \sum_i a_i \frac{\partial}{\partial x_i}$, $a_i \in C^\infty(M)$, then $\overline{X} = \sum_i a_i D_i$, and

$$\mathcal{C}D(\pi) = \left\{ \sum_i \varphi_i D_i \mid \varphi_i \in \mathcal{F}(\pi) \right\}.$$

Vector field $X \in D(\pi)$ is called vertical if $X(\varphi) = 0$ for each $\varphi \in C^\infty(M) \subset \mathcal{F}(\pi)$. Denote by $D^V(\pi) \subset D(\pi)$ the $\mathcal{F}(\pi)$-module of vertical vector fields. D^V is also a subalgebra of the Lie algebra $D(\pi)$,

$$[D^V(\pi), D^V(\pi)] \subset D^V(\pi), \tag{2}$$

and $D(\pi)$ splits into a direct sum

$$D(\pi) = \mathcal{C}D(\pi) \oplus D^V(\pi).$$

Dually, the module of 1-form on $J^\infty(\pi)$ splits into a direct sum

$$\Lambda^1(\pi) = \mathcal{C}^1\Lambda^1(\pi) \oplus \overline{\Lambda}^1(\pi), \tag{3}$$

where

$$\overline{\Lambda}^1(\pi) = \big\{\omega \in \Lambda(\pi) \mid \omega(X) = 0 \text{ for any } X \in D^V(\pi)\big\}$$

is the module of *horizontal* 1*-forms*, and

$$\mathcal{C}^1\Lambda^1(\pi) = \{\omega \in \Lambda(\pi) \mid \omega(X) = 0 \quad \text{for any} \quad X \in \mathcal{C}D(\pi)\}$$

is the module of *Cartan forms.*

Locally,

$$\overline{\Lambda}^1(\pi) = \left\{\sum_i \varphi_i dx_i \mid \varphi_i \in \mathcal{F}(\pi)\right\},$$

$$\mathcal{C}^1\Lambda^1(\pi) = \left\{\sum_{j,\sigma} \varphi_j^\sigma \omega_\sigma^j \mid \varphi_j^\sigma \in \mathcal{F}(pi)\right\},$$

where $\omega_\sigma^j = dp_\sigma^j - \sum_i p_{\sigma+1_i}^j dx_i$.

From (3) it follows that each $\Lambda^i(\pi)$, $i > 0$, splits into

$$\Lambda^i(\pi) = \sum_{\alpha+\beta=i} \overline{\Lambda}^\alpha(\pi) \otimes \mathcal{C}^\beta\Lambda^1(\pi),$$

where

$$\overline{\Lambda}^\alpha(\pi) = \underbrace{\overline{\Lambda}^1(\pi) \wedge \cdots \wedge \overline{\Lambda}^1(\pi)}_{\alpha \text{ times}},$$

$$\mathcal{C}^\beta\Lambda^1(\pi) = \underbrace{\mathcal{C}^1\Lambda^1(\pi) \wedge \cdots \wedge \mathcal{C}^1\Lambda^1(\pi)}_{\beta \text{ times}}.$$

The graded algebra $\overline{\Lambda}^*(\pi)$ is the lifting of the graded algebra $\Lambda^*(M)$ of differential forms on M. The lifting of the de Rham differential $d : \Lambda^i(M) \to \Lambda^{i+1}(M)$ is called *the horizontal differential* and is denoted by $\overline{d} : \overline{\Lambda}^i(\pi) \to \overline{\Lambda}^{i+1}(\pi)$. The complex

$$0 \xrightarrow{\mathcal{F}} (\pi) \xrightarrow{\overline{d}} \overline{\Lambda}^1(\pi) \xrightarrow{\overline{d}} \dots \xrightarrow{\overline{d}} \overline{\Lambda}^n(\pi) \xrightarrow{0}$$

is called *the horizontal de Rham complex* and its cohomology at the term $\overline{\Lambda}^i(\pi)$ is denoted by $\overline{H}^i(\pi)$. In local coordinates

$$\overline{d}(dx_i) = 0, \quad \overline{d}(\varphi) = \sum_i D_i(\varphi)dx_i, \ \varphi \in \mathcal{F}(\pi).$$

Let $V(\pi) : V(E) \to E$ be the vector bundle of vertical vector fields on E, $V(E) = \{v \in T(E) \mid \pi_* v = 0\}$. By definition, put $\varkappa = \overline{\Gamma(V(\pi))} = \Gamma\left(\pi^*_{\infty,0}(V(\pi))\right)$. Then there exists a map

$$\ni\colon \varkappa \to D^V(\pi),\ \varphi \mapsto \ni_\varphi,$$

where $\ni_\varphi$ is called *evolutionary derivation* and is defined by following

$$\ni_\varphi(\psi)|_{[f]^\infty} = \left.\frac{d}{dt}\right|_{t=0} \left(\psi|_{[f_t]^\infty}\right),$$

where $f \in \Gamma(\pi)$, $\varphi \in \varkappa$, $\psi \in \mathcal{F}(\pi)$, and f_t is a 1-parameter family of sections of π such that $\left.\frac{d}{dt}\right|_{t=0} f_t = \varphi|_{[f]^\infty}$, $f_0 = f$.

In local coordinates, if $\varphi = (\varphi^1, \ldots, \varphi^m)$, then

$$\ni_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j)\frac{\partial}{\partial p^j_\sigma}.$$

2.2. **Differential operators and equations.** *The system of nonlinear differential equations* of order k imposed on sections of $\pi : E \to M$ is a submanifold $\mathcal{Y}^k \subset J^k(\pi)$. Denote by $\mathcal{Y}^{k+s} \subset J^{k+s}(\pi)$ the s-th prolongation of $\mathcal{Y}^k$. We will always suppose that $\mathcal{Y}^k$ is formally integrable. Then $\mathcal{Y}^s$, $s \geq k$, is a smooth manifold, and $\pi_{s,t}$, $s \geq t \geq k$, maps $\mathcal{Y}^s$ onto $\mathcal{Y}^t$ surjectively. The inverse limit of the system of maps

$$\pi_{s,t} : \mathcal{Y}^s \to \mathcal{Y}^t, \quad s \geq t \geq k,$$

is called *an infinitely prolonged differential equation*, or simply *a differential equation*, and is denoted by $\mathcal{Y} = \mathcal{Y}^\infty$. Obviously, the infinite jets manifold $J^\infty(\pi)$ is a differential equation of zero order with $\mathcal{Y}^k = J^k(\pi), k \geq 0$.

One can construct a calculus on a differential equation $\mathcal{Y} \subset J^\infty(\pi)$ in the same way as for the jets manifold $J^\infty(\pi)$. Let $\mathcal{F}, \Lambda^*, D$ denote the algebra of smooth functions, the graded algebra of differential forms and the module of vector fields on $\mathcal{Y}$ respectively.

As for the jets manifold $J^\infty(\pi)$, there exists a splitting of the modules of vector fields D and 1-forms Λ^1

$$\begin{aligned} D &= D^V \oplus \mathcal{C}D, \\ \Lambda^1 &= \overline{\Lambda}^1 \oplus \mathcal{C}^1\Lambda^1. \end{aligned} \tag{4}$$

Let $\xi : E' \to M$ be a vector bundle, $F : \Gamma(\pi) \to \Gamma(\xi)$ be a nonlinear differential operator, F can be considered as a section of the induced vector bundle $\pi^*_\infty(\xi)$ over $J^\infty(\pi)$. Define a smooth map

$$J(F) : J^\infty(\pi) \to J^\infty(\xi), [f]^\infty_x \mapsto [F(f)]^\infty_x.$$

Let $\mathcal{Y} = \mathcal{Y}(F)$ be the differential equation defined by F,

$$\mathcal{Y}(F) = \ker J(F) = \{\theta \in J^\infty(\pi) \mid J(F)(\theta) = 0\}.$$

In local coordinates, if $F = (F^1, \dots, F^s)$, $F^l = F^l(x_i, p^j_\sigma)$, then $\mathcal{Y}(F)$ is the infinite prolongation of the system

$$\begin{cases} F^1 = 0, \\ \dots\dots\dots \\ F^s = 0. \end{cases}$$

Denote by $P = \Gamma(\pi^*_\infty(\xi))$ the $\mathcal{F}(\pi)$-module of sections of the induced vector bundle over $J^\infty(\pi)$. The *universal linearization of* F is a $\mathcal{C}$-differential operator $\ell_F \in \mathcal{C}\mathrm{Diff}(\varkappa; P)$ such that

$$\ell_F(\varphi)|_{[f]^\infty} = \left.\frac{d}{dt}\right|_{t=0} F(f_t), \quad \varphi \in \varkappa, f \in \Gamma(\pi),$$

where f_t is a 1-parameter family of sections of π such that $\frac{d}{dt}\big|_{t=0} f_t = \varphi|_{[f]^\infty}$, $f_0 = f$. In local coordinates

$$\ell_F = \begin{pmatrix} \sum_\sigma \frac{\partial F^1}{\partial p^1_\sigma} D_\sigma & \dots & \sum_\sigma \frac{\partial F^1}{\partial p^m_\sigma} D_\sigma \\ \dots\dots\dots & & \dots\dots\dots \\ \sum_\sigma \frac{\partial F^s}{\partial p^1_\sigma} D_\sigma & \dots & \sum_\sigma \frac{\partial F^s}{\partial p^m_\sigma} D_\sigma \end{pmatrix}.$$

Denote by $I_\mathcal{Y} \subset \mathcal{F}(\pi)$ the ideal of functions identically equal to 0 on $\mathcal{Y}$. For each $\mathcal{F}(\pi)$-module R denote by $[R]$ *the restriction of* R *to* $\mathcal{Y}$, $[R] = R/(I_\mathcal{Y} \cdot R)$. If R is a projective $\mathcal{F}(\pi)$-module, then $[R]$ is a projective $\mathcal{F}$-module.

A differential equation $\mathcal{Y} = \mathcal{Y}(F)$ is called *regular* with respect to the nonlinear differential operator $F = (F^1, \dots, F^s)$ if

1. ideal $I_\mathcal{Y}$ is generated by functions $D_\sigma(F^l)$, $1 \le l \le s$, $0 \le |\sigma| < \infty$;
2. the module of 1-forms Λ^1 is projective.

This definition does not depend on the choice of coordinate chart. In the geometrical theory of nonlinear differential equations a regular differential equation is an analog of a smooth (without singularities) manifold.

If the $\mathcal{F}$-module Λ^1 is projective, then from (4) it follows that modules $\overline{\Lambda}^1$ and $\mathcal{C}^1\Lambda^1$ are also projective. From now onwards we assume that all the modules under consideration are projective.

Let $i : \mathcal{Y} \to J^\infty(\pi)$ be natural inclusion, $i^* : [\mathcal{C}^1\Lambda^1(\pi)] \to \mathcal{C}^1\Lambda^1$ be induced homomorphism, $[\ell_F] : [\varkappa] \to [P]$ be the restriction of the universal linearization to $\mathcal{Y}$.

Proposition 1 ([10]). 1. $[\mathcal{C}^1\Lambda^1(\pi)] = \mathcal{C}\mathrm{Diff}([\varkappa]; \mathcal{F})$;

2. *The following sequence is exact*

$$\mathcal{C}\mathrm{Diff}([P]; \mathcal{F}) \xrightarrow{[\ell_F]} \mathcal{C}\mathrm{Diff}([\varkappa]; \mathcal{F}) \xrightarrow{i^*} \mathcal{C}^1\Lambda^1 \xrightarrow{0},$$
$$[\ell_F](\nabla) = \nabla \circ [\ell_F].$$

In local coordinates the isomorphism of statement (1) is given by following

$$\omega_\sigma^j = dp_\sigma^j - \sum_i p_{\sigma+1_i}^j dx_i \mapsto \nabla_\sigma^j, \quad \nabla_\sigma^j(\varphi^1, \dots, \varphi^m) = D_\sigma(\varphi^j).$$

A differential equation $\mathcal{Y} = \mathcal{Y}(F)$ is called *determined* if the homomorphism

$$[\ell_F] : \mathcal{C}\mathrm{Diff}([P]; \mathcal{F}) \to \mathcal{C}\mathrm{Diff}([\varkappa]; \mathcal{F})$$

is an injection. This condition always holds for evolution equations, or for differential equations $\mathcal{Y}(F)$ such that the symbol of the matrix differential operator $\ell_F : \varkappa \to P$ has the maximal rank almost everywhere.

If a differential equation $\mathcal{Y}(F)$ is regular and determined then the short exact sequence

$$0 \xrightarrow{\mathcal{C}\mathrm{Diff}} ([P]; \mathcal{F}) \xrightarrow{[\ell_F]} \mathcal{C}\mathrm{Diff}([\varkappa]; \mathcal{F}) \xrightarrow{i^*} \mathcal{C}^1\Lambda^1 \xrightarrow{0} \tag{5}$$

splits and $\mathcal{C}^1\Lambda^1$ can be considered as a submodule of $\mathcal{C}\mathrm{Diff}([\varkappa]; \mathcal{F})$.

2.3. **Adjoint operator.** Let $\mathcal{Y}$ be a differential equation. For any $\mathcal{F}$-module Q consider the following complex $(\mathcal{S}(Q), \overline{d}_1)$:

$$0 \xrightarrow{\mathcal{C}\mathrm{Diff}} (Q; \mathcal{F}) \xrightarrow{\overline{d}_1} \mathcal{C}\mathrm{Diff}(Q; \overline{\Lambda}^1) \xrightarrow{\overline{d}_1} \dots \xrightarrow{\overline{d}_1} \mathcal{C}\mathrm{Diff}(Q; \overline{\Lambda}^n) \xrightarrow{0},$$
$$\overline{d}_1(\Delta) = -\overline{d} \circ \Delta. \tag{6}$$

The cohomology of complex (6) is described by

Proposition 2 ([10])**.** 1. $H^i(\mathcal{S}(Q)) = 0$ *if* $i \neq n$*;*
2. $H^n(\mathcal{S}(Q)) = \mathrm{Hom}_{\mathcal{F}}(Q; \overline{\Lambda}^n)$.

By definition, put $\hat{Q} = \mathrm{Hom}_{\mathcal{F}}(Q; \overline{\Lambda}^n)$ for any module Q.

Each $\mathcal{C}$-differential operator $\nabla : Q_1 \to Q_2$ induces a homomorphism of complexes $\mathcal{S}(\nabla) : \mathcal{S}(Q_2) \to \mathcal{S}(Q_1)$, $\mathcal{S}(\nabla)(\Delta) = \Delta \circ \nabla$, $\Delta \in \mathcal{C}\mathrm{Diff}(Q_2; \overline{\Lambda}^i)$, and an $\mathbb{R}$-linear map of the cohomology

$$\nabla^* : \hat{Q}_2 \to \hat{Q}_1.$$

The operator ∇^* is called *the adjoint operator* for ∇.

Proposition 3 ([10])**.** 1. $\nabla^* \in \mathcal{C}\mathrm{Diff}(\hat{Q}_2; \hat{Q}_1)$.
2. *For all* $\nabla_1 \in \mathcal{C}\mathrm{Diff}(Q_1; Q_2)$*,* $\nabla_2 \in \mathcal{C}\mathrm{Diff}(Q_2; Q_3)$

$$(\nabla_2 \circ \nabla_1)^* = \nabla_1^* \circ \nabla_2^*.$$

3. *If in local coordinates* ∇ *is a* $m_1 \times m_2$*-matrix*

$$\nabla = (\sum_\sigma a_{ij}^\sigma D_\sigma),$$

then ∇^* *is the* $m_2 \times m_1$*-matrix*

$$\nabla^* = \left(\sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a^\sigma_{ji} \right).$$

2.4. **The $\mathcal{C}$-spectral sequence.** Consider the $\mathcal{C}$-*filtration* in the de Rham complex on an equation $\mathcal{Y}$

$$\Lambda = \mathcal{C}^0\Lambda \supset \mathcal{C}^1\Lambda \supset \mathcal{C}^2\Lambda \supset \dots,$$

where

$$\mathcal{C}^p\Lambda = \sum_{\substack{\alpha \geq p \\ \beta \geq 0}} \mathcal{C}^\alpha\Lambda^1 \otimes \overline{\Lambda}^\beta.$$

The spectral sequence $\{E_r^{p,q}, d_r^{p,q}\}$ determined by this filtration is called *the $\mathcal{C}$-spectral sequence* for the differential equation $\mathcal{Y}$. As usual p denotes the filtration degree and $p+q$ denotes the total degree.

Proposition 4 ([10])**.** 1. $E_0^{p,q} = \mathcal{C}^p\Lambda^1 \otimes \overline{\Lambda}^q$;
2. $E_0^{0,q} = \overline{\Lambda}^q$, $d_0^{0,q} = \overline{d}$;
3. *The $\mathcal{C}$-spectral sequence converges and its term* $E_\infty = \{E_\infty^{p,q}\}$ *is attached to* $H^*_{de\ Rham}(\mathcal{Y})$.

Keeping in mind statement (2) of Proposition 4 denote

$$d_0^{p,q} = \overline{d} : \mathcal{C}^p\Lambda^1 \otimes \overline{\Lambda}^q \to \mathcal{C}^p\Lambda^1 \otimes \overline{\Lambda}^{q+1}.$$

If $\mathcal{Y} = J^\infty(\pi)$, then in local coordinates $\overline{d}$ is uniquely defined by following

$$\overline{d}(f) = \sum_i D_i(f) dx_i, \ f \in \mathcal{F};$$

$$\overline{d}(dx_i) = 0, \ dx_i \in \overline{\Lambda}^1;$$

$$\overline{d}(\omega^j_\sigma) = d\omega^j_\sigma = \sum_i dx_i \wedge \omega^j_{\sigma+1_i}, \ \omega^j_\sigma \in \mathcal{C}^1\Lambda^1.$$

3. The term E_1 of the $\mathcal{C}$-spectral sequence

In this section we suppose that $\mathcal{Y} = \mathcal{Y}(\mathcal{F})$ is a regular differential equation defined by a nonlinear differential operator F.

3.1. **Multilinear $\mathcal{C}$-differential operators.** Let $Q_1, \dots, Q_p, R$ be $\mathcal{F}$-modules. Consider the module of $\mathbb{R}$-multilinear $\mathcal{C}$-differential operators $\Delta : Q_1 \times \dots \times Q_p \to R$

$$\begin{aligned}\mathcal{C}\mathrm{Diff}(Q_1, \dots, Q_p; R) &= \mathcal{C}\mathrm{Diff}\,(Q_1; \mathcal{C}\mathrm{Diff}\,(Q_2; \dots \mathcal{C}\mathrm{Diff}(Q_p; R) \dots)) \\ &= \mathcal{C}\mathrm{Diff}(Q_1; \mathcal{F}) \otimes \dots \otimes \mathcal{C}\mathrm{Diff}(Q_p; \mathcal{F}) \otimes R.\end{aligned}$$

For any $\mathcal{F}$-modules Q, R, by definition, put

$$\mathcal{C}\mathrm{Diff}_p(Q; R) = \mathcal{C}\mathrm{Diff}(\underbrace{Q, \dots, Q}_{p \text{ times}}; R) = \bigotimes^p \mathcal{C}\mathrm{Diff}(Q; \mathcal{F}) \otimes R,$$

and consider the submodules $\mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}(Q;R) \subset \mathcal{C}\mathrm{Diff}_p(Q;R)$

$$\mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}(Q;R) = \bigwedge^p \mathcal{C}\mathrm{Diff}(Q;\mathcal{F}) \otimes R.$$

Module $\mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}(Q;R)$ can be identified with the module of skew-symmetric $\mathcal{C}$-differential operators $\Delta : \underbrace{Q \times \cdots \times Q}_{p \text{ times}} \to R$.

Let $\nabla : R_1 \to R_2$ be a $\mathcal{C}$-differential operator. For each $\mathcal{F}$-module Q define an operator

$$\nabla_p : \mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}(Q;R_1) \to \mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}(Q;R_2),$$
$$\Delta \mapsto (-1)^p \nabla \circ \Delta.$$

Proposition 5 ([10])**.** *Let $\{E_0^{p,q}(\pi), d_0^{p,q}(\pi)\}$ be the zero term of the $\mathcal{C}$-spectral sequence for $J^\infty(\pi)$. Then*

$$E_0^{p,q}(\pi) = \mathcal{C}^p\Lambda^1(\pi) \otimes \overline{\Lambda}^q(\pi) = \mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}\left(\varkappa; \overline{\Lambda}^q(\pi)\right);$$
$$d_0^{p,q}(\pi) = (\overline{d})_p,$$

where $\overline{\mathrm{d}} : \overline{\Lambda}^q(\pi) \to \overline{\Lambda}^{q+1}(\pi)$ is the horizontal differential on $J^\infty(\pi)$.

Lemma 1. 1. *Let $\{[E_0^{p,q}(\pi)], [d_0^{p,q}(\pi)]\}$ be the restriction of the zero term of the $\mathcal{C}$-spectral sequence for $J^\infty(\pi)$ to the equation $\mathcal{Y}$. Then*

$$[E_0^{p,q}(\pi)] = \left[\mathcal{C}^p\Lambda^1(\pi)\right] \otimes \overline{\Lambda}^q = \mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}\left([\varkappa]; \overline{\Lambda}^q\right);$$
$$[d_0^{p,q}(\pi)] = (\overline{d})_p,$$

where $\overline{\mathrm{d}} : \overline{\Lambda}^q \to \overline{\Lambda}^{q+1}$ is the horizontal differential.

2. *For each $\mathcal{F}$-module Q the following sequence is exact*

$$\mathcal{C}\mathrm{Diff}\left([P];\mathcal{F}\right) \otimes \mathcal{C}\mathrm{Diff}_{p-1}^{\mathrm{alt}}\left([\varkappa];Q\right) \xrightarrow{\ell} \mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}\left([\varkappa];Q\right) \xrightarrow{i^*} \mathcal{C}^p\Lambda^1 \otimes Q \xrightarrow{0},$$

where

$$\ell(\Delta_1 \otimes \Delta_2) = (\Delta_1 \circ [\ell_F]) \wedge \Delta_2, \quad \Delta_1 \in \mathcal{C}\mathrm{Diff}([P];\mathcal{F}), \Delta_2 \in \mathcal{C}\mathrm{Diff}_{p-1}^{\mathrm{alt}}([\varkappa];Q).$$

3. *If $\nabla : Q_1 \to Q_2$ is a $\mathcal{C}$-differential operator, then there exists a unique $\mathcal{C}$-differential operator $\nabla_p : \mathcal{C}^p\Lambda^1 \otimes Q_1 \to \mathcal{C}^p\Lambda^1 \otimes Q_2$ such that the following diagram is commutative*

$$\begin{array}{ccc}
\mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}\left([\varkappa];Q_1\right) & \xrightarrow{\nabla_p} & \mathcal{C}\mathrm{Diff}_p^{\mathrm{alt}}\left([\varkappa];Q_2\right) \\
{\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow \\
\mathcal{C}^p\Lambda^1 \otimes Q_1 & \xrightarrow{\nabla_p} & \mathcal{C}^p\Lambda^1 \otimes Q_2
\end{array}$$

4. *$d_0^{p,q} = (\overline{d})_p$, where $d_0^{p,q}$ denotes the zero differential in the $\mathcal{C}$-spectral sequence for $\mathcal{Y}$.*

Proof. Statement (1) follows from Proposition 5. Statement (2) follows from Proposition 1. Statement (3) follows from (2). Finally, (4) follows from (1), (3), and the fact that the following diagram is commutative

$$\begin{array}{ccc} [E_0^{p,q}(\pi)] & \xrightarrow{[d_0^{p,q}(\pi)]} & [E_0^{p,q+1}(\pi)] \\ {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow \\ E_0^{p,q} & \xrightarrow{d_0^{p,q}} & E_0^{p,q+1} \end{array}$$

□

The action of $\mathcal{C}$-differential operators on the module $\mathcal{C}^p\Lambda^1 \otimes Q_1$ of differential forms with coefficients in Q_1 defined by Lemma 1 coincides with the one introduced in [4].

Let $K = \{K^i, \delta^i\}$ be a complex such that δ^i for any i is a $\mathcal{C}$-differential operator. By Lemma 1 we have a complex $\mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p; K) = \{\mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p; K^i), \delta^i_{1\ldots1}\}$ for any modules $R_1, \ldots, R_p$, and a complex $\mathcal{C}^p\Lambda^1 \otimes K = \{\mathcal{C}^p\Lambda^1 \otimes K^i, \delta^i_p\}$ for any integer $p \geq 0$.

Proposition 6 ([10]). 1. *Let $R_1, \ldots, R_p, Q$ be $\mathcal{F}$-modules. Then*

$$H^i\left(\mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p; \mathcal{S}(Q))\right) = \begin{cases} 0 & , i \neq n; \\ \mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p; \hat{Q}) & , i = n. \end{cases}$$

2. *Let $\nabla : Q_1 \to Q_2$ be a $\mathcal{C}$-differential operator. Then*

$$(\nabla^*)_{1\ldots1} : \mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p; \hat{Q}_2) \to \mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p; \hat{Q}_1)$$

is the induced map of the cohomology.

Let $\nabla \in \mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p; Q)$. Take $r_i \in R_i$, $i = 2, \ldots, p$, and consider a $\mathcal{C}$-differential operator

$$\nabla^{(1)}_{r_2,\ldots,r_p} : R_1 \to Q, \quad \nabla^{(1)}_{r_2,\ldots,r_p}(r) = \nabla(r, r_2, \ldots, r_p),$$

where $r \in R_1$. A $\mathcal{C}$-differential operator

$$\nabla^{*_1} : \hat{Q} \times R_2 \times \cdots \times R_p \to \hat{R}_1, \quad \nabla^{*_1}(\hat{q}, r_2, \ldots, r_p) = \left(\nabla^{(1)}_{r_2,\ldots,r_p}\right)^*(\hat{q}),$$

where $\hat{q} \in \hat{Q}$, $r_i \in R_i$, is called *adjoint to ∇ with respect to the 1-st argument*. In the same manner one can define for any $s = 1, \ldots, p$ an operator ∇^{*_s} adjoint to ∇ with respect to the s-th argument.

Denote by $\langle \cdot, \cdot \rangle : Q \times \hat{Q} \to \overline{\Lambda}^n$ the natural pairing, $\langle q, \varphi \rangle = \varphi(q)$, where $q \in Q$, $\varphi \in \hat{Q} = \mathrm{Hom}_{\mathcal{F}}(Q; \overline{\Lambda}^n)$.

Proposition 7 ([10], the Green formula). *Let $\nabla \in \mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p; \hat{Q})$. Then there exists a $\mathcal{C}$-differential operator $G \in \mathcal{C}\mathrm{Diff}(R_1, \ldots, R_p, Q; \overline{\Lambda}^{n-1})$ such that for any $r_i \in R_i$, $1 \leq i \leq p$, and $q \in Q$*

$$\langle \nabla(r_1, \ldots, r_p), q \rangle - \langle r_1, \nabla^{*_1}(q, r_2, \ldots, r_p) \rangle = (\bar{d}_{1\ldots1}G)(r_1, \ldots, r_p, q). \tag{7}$$

The following Lemma is a main tool for calculating the $\mathcal{C}$-spectral sequence. For more general discussion see [2].

Lemma 2. 1. *Let Q be a $\mathcal{F}$-module. Then*

$$H^i(\mathcal{C}^p\Lambda^1 \otimes \mathcal{S}(Q)) = \begin{cases} 0 & , i \neq n; \\ \mathcal{C}^p\Lambda^1 \otimes \hat{Q} & , i = n. \end{cases}$$

2. *Let $\nabla : Q_1 \to Q_2$ be a $\mathcal{C}$-differential operator. Then $(\nabla^*)_p : \mathcal{C}^p\Lambda^1 \otimes \hat{Q}_2 \to \mathcal{C}^p\Lambda^1 \otimes \hat{Q}_1$ is the induced map of the cohomology.*

Proof. Consider a natural filtration in the complex $\{\mathcal{S}(Q), \overline{d}_1\}$ by the modules

$$F^k = \{\Delta \in \mathcal{C}\mathrm{Diff}(Q; \overline{\Lambda}^i) \mid \mathrm{ord}\,\Delta \leq k + i - n\}, \tag{8}$$

where $\mathrm{ord}\,\Delta$ denotes the order of a $\mathcal{C}$-differential operator Δ. Clearly, $F^0 \subset F^1 \subset \cdots \subset F^k \subset \ldots$, and $\overline{d}_1(F^k) \subset F^k$ for any $k \geq 0$. Let $\{F_r^{k,q}, \delta_r^{k,q}\}$ be the spectral sequence for the complex $\mathcal{S}(Q)$ with respect to filtration (8). It is easy to see that

$$F_0^{k,q} = Q^* \otimes S^{2k+q-n}\overline{\Lambda}^{1*} \otimes \overline{\Lambda}^{k+q} = Q^* \otimes S^{2k+q-n}V^* \otimes \Lambda^{n-k-q}V^* \otimes \overline{\Lambda}^n,$$

where $V = \overline{\Lambda}^1$, if $0 \leq k+q \leq n$, $2k+q-n \geq 0$, and $F_0^{k,q} = 0$ otherwise. Further, the zero differential $\delta_0^{k,q}$ has a simple form $\delta_0^{k,q} = 1 \otimes \delta^k \otimes 1$, where

$$\delta^k : S^iV^* \otimes \Lambda^{k-i}V^* \to S^{i+1}V^* \otimes \Lambda^{k-i-1}V^*$$

is the Koszul differential. Hence, $F_1^{k,q} = 0$ if $k \neq 0$ or $q \neq n$, and $F_1^{0,n} = Q^* \otimes \overline{\Lambda}^n = \hat{Q}$.

Filtration (8) of the complex $\mathcal{S}(Q)$ yields a filtration of the complex $\mathcal{C}^p\Lambda^1 \otimes \mathcal{S}(Q)$ by modules $\tilde{F}^k = \mathcal{C}^p\Lambda^1 \otimes F^k$. The corresponding spectral sequence $\{\tilde{F}_r^{k,q}, \tilde{\delta}_r^{k,q}\}$ has the zero term $\tilde{F}_0^{k,q} = \mathcal{C}^p\Lambda^1 \otimes F_0^{k,q}$ and $\tilde{\delta}_0^{k,q} = 1 \otimes \delta_0^{k,q}$. Hence, $\tilde{F}_1^{k,q} = 0$ if $k \neq 0$ or $q \neq n$, and $\tilde{F}_1^{0,n} = \mathcal{C}^p\Lambda^1 \otimes \hat{Q}$. This concludes the proof of statement (1). Statement (2) follows now from Proposition 6 and Lemma 1. □

3.2. **The term E_1 for determined equations.** In this subsection we suppose that $\mathcal{Y} = \mathcal{Y}(F)$ is a regular determined equation.

Lemma 3. *Let H^i be the cohomology of the complex $\left\{\mathcal{C}^{p-1}\Lambda^1 \otimes \mathcal{C}^1\Lambda^1 \otimes \overline{\Lambda}^*, \overline{d}_{1,p-1}\right\}$. Then*

$$H^i = \begin{cases} 0, & i \neq n-1, n, \\ \ker\,[\ell_F]_{p-1}^*, & i = n-1, \\ \mathrm{coker}\,[\ell_F]_{p-1}^*, & i = n, \end{cases} \qquad [\ell_F]_{p-1}^* : \mathcal{C}^{p-1}\Lambda^1 \otimes [\hat{P}] \to \mathcal{C}^{p-1}\Lambda^1 \otimes [\hat{\varkappa}].$$

Proof. Since the equation $\mathcal{Y}$ is regular and determined, sequence (5) is exact. Consider the following short exact sequence of complexes

$$0 \xrightarrow{\mathcal{C}^{p-1}} \Lambda^1 \otimes \mathcal{S}([P]) \xrightarrow{1\otimes\mathcal{S}([\ell_F])} \mathcal{C}^{p-1}\Lambda^1 \otimes \mathcal{S}([\varkappa]) \xrightarrow{1\otimes i^*} \mathcal{C}^{p-1}\Lambda^1 \otimes \mathcal{C}^1\Lambda^1 \otimes \overline{\Lambda}^* \xrightarrow{0}.$$

By Lemma 2 the long exact sequence of the cohomology has the form

$$0 \xrightarrow{H}{}^{n-1} \xrightarrow{\mathcal{C}}{}^{p-1} \Lambda^1 \otimes [\hat{P}] \xrightarrow{[\ell_F]^*_{p-1}} \mathcal{C}^{p-1}\Lambda^1 \otimes [\hat{\varkappa}] \xrightarrow{H}{}^{n} \xrightarrow{0},$$

and the Lemma is obvious. □

The permutation group S_p acts in the complex $\left\{\mathcal{C}\mathrm{Diff}_p([\varkappa];\Lambda^*), \overline{d}_p\right\}$. For each $\tau \in S_p$ and $\nabla \in \mathcal{C}\mathrm{Diff}_p([\varkappa];\Lambda^q)$ we have

$$\tau(\nabla)(\chi_1, \dots, \chi_p) = \nabla(\chi_{\tau(1)}, \dots, \chi_{\tau(p)}). \tag{9}$$

It is easy to see that action (9) projects to an action of the group S_p in the complex $\left\{\mathcal{C}^{p-1}\Lambda^1 \otimes \mathcal{C}^1\Lambda^1 \otimes \overline{\Lambda}^*, \overline{d}_{1,p-1}\right\}$ and, therefore, induces an action in the cohomology. The complex $\mathcal{C}^p\Lambda^1 \otimes \overline{\Lambda}^* = E_0^{p,*}$ is the antisymmetric part of the complex $\mathcal{C}^{p-1}\Lambda^1 \otimes \mathcal{C}^1\Lambda^1 \otimes \overline{\Lambda}^*$ with respect to the action of S_p. Hence, the cohomology $E_1^{p,q} = H^q(\mathcal{C}^p\Lambda^1 \otimes \overline{\Lambda}^*)$ is the antisymmetric part of $H^q\left(\mathcal{C}^{p-1}\Lambda^1 \otimes \mathcal{C}^1\Lambda^1 \otimes \overline{\Lambda}^*\right)$. Combining this with Lemma 3 one immediately obtain the following

Theorem 1 ([10],the two line theorem)**.** *Let $\mathcal{Y}$ be a regular determined differential equation. Then*

1. $E_1^{p,q} = 0,\ p \geq 1, q \neq n-1, n;$
2. $E_1^{p,n-1}$ *(resp. $E_1^{p,n}$) is the antisymmetric part of* $\ker[\ell_F]^*_{p-1}$ *(resp.* $\mathrm{coker}[\ell_F]^*_{p-1}$*) with respect to the induced action of the permutation group S_p.*

Let us now describe the action of the permutation group S_p in $\ker\,[\ell_F]^*_{p-1}$ and $\mathrm{coker}\,[\ell_F]^*_{p-1}$. Denote $\ell = [\ell_F]$, $\ell_{p-1} = [\ell_F]_{p-1}$.

Lemma 4. *Let $\omega \in \ker\,\ell^*_{p-1} \subset \mathcal{C}^{p-1}\Lambda^1 \otimes [\hat{P}]$ and $\nabla = \nabla_\omega \in \mathcal{C}\mathrm{Diff}_{p-1}([\varkappa];[\hat{P}])$ be a $\mathcal{C}$-differential operator such that $i^*(\nabla) = \omega$. Then the following is true.*

1. *There exist operators $\Delta_s \in \mathcal{C}\mathrm{Diff}(\underbrace{[\varkappa],\dots}_{s-1},[P],\underbrace{\dots,[\varkappa]}_{p-s-1};[\hat{\varkappa}])$, $1 \leq s \leq p-1$, such that*
$$\ell^*\left(\nabla(\chi_1, \dots, \chi_{p-1})\right) = \sum_{s=1}^{p-1} \Delta_s(\chi_1, \dots, \ell(\chi_s), \dots, \chi_{p-1}). \tag{10}$$
2. *If $\tau \in S_{p-1} \subset S_p$, $\tau(p) = p$, then*
$$\tau(\omega) = i^*(\tau(\nabla)),$$
where $\tau(\nabla)$ is defined by (9).
3. *If $\tau = (s,p)$ is a transposition that interchanges s and p then*
$$\tau(\omega) = -i^*\left(\Delta_s^{*_s}\right),$$
where Δ^{*_s} denotes the adjoint operator for Δ with respect to the s-th argument.

Proof. Statement (1) follows from the definition of ℓ_{p-1}.

In order to prove statements (2) and (3) let us describe the isomorphism between ker ℓ^*_{p-1} and $H^{n-1}(\mathcal{C}^1\Lambda^1 \otimes \mathcal{C}^{p-1}\Lambda^1 \otimes \overline{\Lambda}^*)$ given by Lemma 2. If $\omega \in$ ker ℓ^*_{p-1} and $\omega = i^*(\nabla)$, where $\nabla \in \mathcal{C}\mathrm{Diff}_{p-1}([\varkappa]; [\hat{P}])$, then by Green formula (7) applied to the operator ℓ there exists a $\mathcal{C}$-differential operator $A \in \mathcal{C}\mathrm{Diff}_p([\varkappa]; \overline{\Lambda}^{n-1})$ such that

$$\langle \ell^* \nabla(\chi_1, \ldots, \chi_{p-1}), \chi_p \rangle - \langle \nabla(\chi_1, \ldots, \chi_{p-1}), \ell \chi_p \rangle = (\overline{d}_p A)(\chi_1, \ldots, \chi_p).$$

The cohomology class corresponding to ω is defined by a differential form $\eta = i^*(A) \in \mathcal{C}^1\Lambda^1 \otimes \mathcal{C}^{p-1}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$.

Now statement 2 is obvious. Indeed, if $\tau \in S_{p-1} \subset S_p$, then

$$\langle \ell^* \tau(\nabla)(\chi_1, \ldots, \chi_{p-1}), \chi_p \rangle - \langle \tau(\nabla)(\chi_1, \ldots, \chi_{p-1}), \ell \chi_p \rangle = (\overline{d}_p \tau(A))(\chi_1, \ldots, \chi_p),$$

and the cohomology class corresponding to $\tau(\nabla)$ is defined by a differential form $\eta' = i^*(\tau(A)) = \tau(i^*(A)) = \tau(\eta)$.

Let us prove statement (3). Without loss of generality assume that τ is a transposition $(1, p)$. From (10) we have

$$\ell^* \Delta_1^{*_1}(\chi_1, \ldots, \chi_{p-1}) = \nabla^{*_1}(\ell \chi_1, \ldots, \chi_{p-1}) - \sum_{k=2}^{p-1} \Delta_k^{*_1}(\chi_1, \ldots, \ell \chi_k, \ldots, \chi_{p-1}). \tag{11}$$

A $\mathcal{C}$-differential operator $-\Delta_1^{*_1}$ defines a cohomology class $\eta' = i^*(B)$, where $B \in \mathcal{C}\mathrm{Diff}_p([\varkappa]; \overline{\Lambda}^{n-1})$ satisfies the following equation

$$(\overline{d}_p B)(\chi_1, \ldots, \chi_p) = -\langle \ell^* \Delta_1^{*_1}(\chi_1, \ldots, \chi_{p-1}), \chi_p \rangle + \langle \Delta_1^{*_1}(\chi_1, \ldots, \chi_{p-1}), \ell \chi_p \rangle.$$

But from (11) we have

$$
\begin{aligned}
&-\langle \ell^* \Delta_1^{*_1}(\chi_1, \ldots, \chi_{p-1}), \chi_p \rangle + \langle \Delta_1^{*_1}(\chi_1, \ldots, \chi_{p-1}), \ell \chi_p \rangle = \\
&- \langle \nabla^{*_1}(\ell \chi_1, \ldots, \chi_{p-1}), \chi_p \rangle + \\
&\sum_{k=2}^{p-1} \langle \Delta_k^{*_1}(\chi_1, \ldots, \ell \chi_k, \ldots, \chi_{p-1}), \chi_p \rangle + \langle \Delta_1^{*_1}(\chi_1, \ldots, \chi_{p-1}), \ell \chi_p \rangle = \\
&- \langle \ell \chi_1, \nabla(\chi_p, \chi_2, \ldots, \chi_{p-1}) \rangle + (\overline{d}_p C_1)(\ell \chi_1, \ldots, \chi_p) \\
&+ \sum_{k=2}^{p-1} \langle \chi_1, \Delta_k(\chi_p, \chi_2, \ldots, \ell \chi_k, \ldots, \chi_{p-1}) \rangle + \sum_{k=2}^{p-1} (\overline{d}_p C_k)(\chi_1, \ldots, \ell \chi_k, \ldots, \chi_p) \\
&+ \langle \chi_1, \Delta_1(\ell \chi_p, \chi_2, \ldots, \chi_{p-1}) \rangle + (\overline{d}_p C_p)(\chi_1, \ldots, \ell \chi_p) = \\
&- \langle \ell \chi_1, \nabla(\chi_p, \chi_2, \ldots, \chi_{p-1}) \rangle \\
&+ \langle \chi_1, (\ell^* \nabla)(\chi_p, \chi_2, \ldots, \chi_{p-1}) \rangle + \sum_{k=1}^{p} (\overline{d}_p C_k)(\chi_1, \ldots, \ell \chi_k, \ldots, \chi_p) =
\end{aligned}
$$

$$(\overline{d}_p\tau(A))(\chi_1,\dots,\chi_p)+\sum_{k=1}^{p}(\overline{d}_pC_k)(\chi_1,\dots,\ell\chi_k,\dots,\chi_p),$$

where the $\mathcal{C}$-differential operators C_k, $k=1,\dots,p$, are defined by Green formula (7). Hence, one can choose the operator B such that

$$B(\chi_1,\dots,\chi_p)=\tau(A)(\chi_1,\dots,\chi_p)+\sum_{k=1}^{p}C_k(\chi_1,\dots,\ell\chi_k,\dots,\chi_p).$$

But, obviously, $i^*(B)=i^*(\tau(A))$, and the operator $-\Delta_1^{*_1}$ corresponds to the cohomology class $\tau(\eta)$ □

Proposition 8 ([10]). *The action of the permutation group S_p in the n-th cohomology group $\mathcal{C}\mathrm{Diff}_{p-1}([\varkappa];[\hat{\varkappa}])$ can be described as follows.*

1. *If $\tau\in S_{p-1}\subset S_p$, $\tau(p)=p$, then*
$$\tau(\nabla)(\chi_1,\dots,\chi_{p-1})=\nabla(\chi_{\tau(1)},\dots,\chi_{\tau(p-1)}),$$
2. *if $\tau=(s,p)$ is a transposition that interchanges s and p, then*
$$\tau(\nabla)=\nabla^{*_s},$$
where $\nabla\in\mathcal{C}\mathrm{Diff}_{p-1}([\varkappa];[\hat{\varkappa}])$.

Corollary 1. *Let $\omega\in\operatorname{coker}\,\ell^*_{p-1}$ and $\nabla=\nabla_\omega\in\mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{p-1}([\varkappa];[\hat{\varkappa}])$ be a $\mathcal{C}$-differential operator such that $\omega\equiv i^*(\nabla)\mod\operatorname{im}\,\ell^*$. Then the following is true.*

1. *If $\tau\in S_{p-1}\subset S_p$, $\tau(p)=p$, then*
$$\tau(\omega)\equiv i^*(\tau(\nabla))\mod\operatorname{im}\,\ell^*,$$
where $\tau(\nabla)$ is defined by (9).
2. *If $\tau=(s,p)$ is a transposition that interchanges s and p then*
$$\tau(\omega)\equiv i^*\left(\nabla^{*_s}\right)\mod\operatorname{im}\,\ell^*.$$

4. The $\mathcal{C}$-spectral sequence for evolution equations

4.1. Let $\mathcal{Y}$ be an evolution equation represented locally as

$$\mathcal{Y}=\left\{u_t-f(x_i,t,u^j_\sigma)=0\right\},\ k=1,\dots,m,$$

where $x_1,\dots,x_{n-1},t$ are independent variables, $u=(u^1,\dots,u^m)$ are dependent variables and

$$u^j_\sigma=\frac{\partial^{|\sigma|}u^j}{\partial x_1^{\sigma_1}\dots\partial x_{n-1}^{\sigma_{n-1}}},\ \sigma=(\sigma_1,\dots,\sigma_{n-1}).$$

Functions x_i,t,p^j_σ form a system of local coordinates on $\mathcal{Y}$. Total derivatives D_t,D_i have the following form

$$D_t=\frac{\partial}{\partial t}+\sum_{j,\sigma}D_\sigma(f^j)\frac{\partial}{\partial p^j_\sigma},\ D_i=\frac{\partial}{\partial x_i}+\sum_{j,\sigma}p^j_{\sigma+1_i}\frac{\partial}{\partial p^j_\sigma},i=1,\dots,n-1,$$

and the universal linearization is a $m \times m$ matrix

$$\ell = [\ell_F] = D_t - L_f = \begin{pmatrix} D_t & 0 & \dots & 0 \\ 0 & D_t & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & D_t \end{pmatrix} - \begin{pmatrix} \sum_\sigma \frac{\partial f^1}{\partial p^1_\sigma} D_\sigma & \dots & \sum_\sigma \frac{\partial f^1}{\partial p^m_\sigma} D_\sigma \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \sum_\sigma \frac{\partial f^m}{\partial p^1_\sigma} D_\sigma & \dots & \sum_\sigma \frac{\partial f^m}{\partial p^m_\sigma} D_\sigma \end{pmatrix}.$$

By definition, put $\mathcal{F}^m = \underbrace{\mathcal{F} \oplus \dots \oplus \mathcal{F}}_{m \text{ times}}$. Then in local coordinates one can identify $[\varkappa] = [P] = [\hat{\varkappa}] = [\hat{P}] = \mathcal{F}^m$. Denote by $\mathcal{C}\mathrm{Diff}_x(\mathcal{F};\mathcal{F})$ the submodule of $\mathcal{C}\mathrm{Diff}(\mathcal{F};\mathcal{F})$ generated by operators D_σ, $\sigma = (\sigma_1, \dots, \sigma_{n-1})$. For any $\mathcal{F}$-modules Q_1, Q_2 by definition put $\mathcal{C}\mathrm{Diff}_x(Q_1;Q_2) = \mathrm{Hom}_\mathcal{F}(Q_1;Q_2) \otimes \mathcal{C}\mathrm{Diff}_x(\mathcal{F};\mathcal{F}) \subset \mathcal{C}\mathrm{Diff}(Q_1;Q_2) = \mathrm{Hom}_\mathcal{F}(Q_1;Q_2) \otimes \mathcal{C}\mathrm{Diff}(\mathcal{F};\mathcal{F})$, and define a linear projection $i^* : \mathcal{C}\mathrm{Diff}(\mathcal{F}^m;\mathcal{F}) \to \mathcal{C}\mathrm{Diff}_x(\mathcal{F}^m;\mathcal{F})$ by following

$$i^*(\nabla) = \nabla, \text{ if } \nabla \in \mathcal{C}\mathrm{Diff}_x(\mathcal{F}^m;\mathcal{F}),$$
$$i^*(\nabla \circ D_t) = i^*(\nabla \circ L_f).$$

Lemma 5. *The sequence*

$$0 \xrightarrow{\mathcal{C}\mathrm{Diff}} (\mathcal{F}^m;\mathcal{F}) \xrightarrow{\ell} \mathcal{C}\mathrm{Diff}(\mathcal{F}^m;\mathcal{F}) \xrightarrow{i^*} \mathcal{C}\mathrm{Diff}_x(\mathcal{F}^m;\mathcal{F}) \xrightarrow{0},$$

where $\ell(\Delta) = \Delta \circ \ell$ is exact.

Proof. Obvious. □

Corollary 2. *If $\mathcal{Y}$ is an evolution equation,then*
1. *$\mathcal{Y}$ is regular and determined;*
2. *$\mathcal{C}^1\Lambda^1 = \mathcal{C}\mathrm{Diff}_x(\mathcal{F}^m;\mathcal{F})$;*

Each multilinear $\mathcal{C}$-differential operator $\Delta \in \mathcal{C}\mathrm{Diff}_{x,p}(\mathcal{F}^m;\mathcal{F}^m)$ can be written in the form

$$\Delta = \left(\Delta_{ij_1\dots j_p}\right) = \left(\sum_{\sigma^1,\dots,\sigma^p} a^{\sigma^1\dots\sigma^p}_{ij_1\dots j_p} D_{\sigma^1} \otimes \dots \otimes D_{\sigma^p} \right),$$
$$\left(\Delta(\chi_1,\dots,\chi_p)\right)^i = \sum_{\substack{\sigma^1,\dots,\sigma^p \\ i,j_1,\dots,j_p}} a^{\sigma^1\dots\sigma^p}_{ij_1\dots j_p} D_{\sigma^1}(\chi_1^{j_1}) \cdot \dots \cdot D_{\sigma^p}(\chi_p^{j_p}),$$

where $1 \le i, j_1, \dots, j_p \le m$, $\sigma^s = (\sigma^s_1, \dots, \sigma^s_{n-1})$, $\chi_s = (\chi^1_s, \dots, \chi^m_s) \in \mathcal{F}^m$.

For each $\nabla \in \mathcal{C}\mathrm{Diff}(\mathcal{F};\mathcal{F})$ and $\Delta \in \mathcal{C}\mathrm{Diff}_{x,p}(\mathcal{F}^m;\mathcal{F}^m)$ define a $\mathcal{C}$-differential operator $\nabla(\Delta) \in \mathcal{C}\mathrm{Diff}_{x,p}(\mathcal{F}^m;\mathcal{F}^m)$ such that

$$(\nabla(\Delta))_{ij_1\dots j_p} = \sum_{\sigma^1,\dots,\sigma^p} \nabla(a^{\sigma^1\dots\sigma^p}_{ij_1\dots j_p}) D_{\sigma^1} \otimes \dots \otimes D_{\sigma^p}.$$

Theorem 2. *The term $E_1^{p,n-1}$, $p \ge 2$, for an evolution equation $\mathcal{Y}$ consists of $\mathcal{C}$-differential operators $\nabla \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{x,p-1}(\mathcal{F}^m;\mathcal{F}^m)$ such that*

1. $\nabla^{*_s} = -\nabla$, $1 \le s \le p-1$;
2. *for any* $\chi_1, \dots, \chi_{p-1} \in \mathcal{F}^m$

$$D_t(\nabla)(\chi_1, \dots, \chi_{p-1}) + L_f^* \left(\nabla(\chi_1, \dots, \chi_{p-1})\right) + \sum_{s=1}^{p-1} \nabla \left(\chi_1, \dots, L_f(\chi_s), \dots, \chi_{p-1}\right) = 0. \tag{12}$$

Proof. By Theorem 1 if $\nabla \in E_1^{p,n-1}$, then there exist operators $\Delta_1, \dots, \Delta_{p-1}$ such that

$$\ell^* \left(\nabla(\chi_1, \dots, \chi_{p-1})\right) = \sum_{s=1}^{p-1} \Delta_s(\chi_1, \dots, \ell(\chi_s), \dots, \chi_{p-1}).$$

But we have

$$-\ell^* \left(\nabla(\chi_1, \dots, \chi_{p-1})\right) = \left(D_t + L_f^*\right) \left(\nabla(\chi_1, \dots, \chi_{p-1})\right) =$$

$$\sum_{s=1}^{p-1} \nabla(\chi_1, \dots, D_t(\chi_s), \dots, \chi_{p-1}) + D_t(\nabla)(\chi_1, \dots, \chi_{p-1}) + L_f^* \left(\nabla(\chi_1, \dots, \chi_{p-1})\right) =$$

$$\sum_{s=1}^{p-1} \nabla(\chi_1, \dots, \ell(\chi_s), \dots, \chi_{p-1}) + D_t(\nabla)(\chi_1, \dots, \chi_{p-1}) + L_f^* \left(\nabla(\chi_1, \dots, \chi_{p-1})\right) + \sum_{s=1}^{p-1} \nabla(\chi_1, \dots, L_f(\chi_s), \dots, \chi_{p-1}).$$

Therefore,

$$D_t(\nabla)(\chi_1, \dots, \chi_{p-1}) + L_f^* \left(\nabla(\chi_1, \dots, \chi_{p-1})\right) + \sum_{s=1}^{p-1} \nabla \left(\chi_1, \dots, L_f(\chi_s), \dots, \chi_{p-1}\right) = 0,$$

and $\Delta_s = -\nabla$. Hence, if $\tau = (s,p) \in S_p$, then $\tau(\nabla) = -\Delta_s^{*_s}$, and $\tau(\nabla) = -\nabla$ if and only if $\nabla = -\nabla^{*_s}$. □

In any module $\mathcal{C}\mathrm{Diff}(Q;R)$ there exists a filtration by the modules $\mathcal{C}\mathrm{Diff}^{(k)}(Q;R)$ consisting of $\mathcal{C}$-differential operators of order $\le k$. Consider *the module of* $\mathcal{C}$*-symbols*

$$\mathcal{C}\mathrm{smbl}(Q;R) = \sum_{k=0}^{\infty} \mathcal{C}\mathrm{smbl}^{(k)}(Q;R),$$

$$\mathcal{C}\mathrm{smbl}^{(k)}(Q;R) = \mathcal{C}\mathrm{Diff}^{(k)}(Q;R)/\mathcal{C}\mathrm{Diff}^{(k-1)}(Q;R).$$

By definition, one has projections

$$\mathcal{C}\mathrm{smbl}^{(k)} : \mathcal{C}\mathrm{Diff}^{(k)}(Q;R) \to \mathcal{C}\mathrm{smbl}^{(k)}(Q;R).$$

For each $\Delta \in \mathcal{C}\mathrm{Diff}(Q;R)$, $\Delta \ne 0$, define the *order* of Δ

$$\operatorname{ord} \Delta = \min\{k \mid \Delta \in \mathcal{C}\mathrm{Diff}^{(k)}(Q;R)\},$$

and the $\mathcal{C}$-*symbol* $s(\Delta) = \mathcal{C}\mathrm{smbl}^{(\mathrm{ord}\,\Delta)}(\Delta)$. Symbols of $\mathcal{C}$-differential operators from $\mathcal{C}\mathrm{Diff}_x(Q;R)$ generate a submodule $\mathcal{C}\mathrm{smbl}_x(Q;R) \in \mathcal{C}\mathrm{smbl}(Q;R)$.

By definition, put $s(D_i) = \theta_i$, $s(D_t) = \theta_t$, then $\theta_i, \theta_t \in \mathcal{C}\mathrm{smbl}^{(1)}(\mathcal{F};\mathcal{F})$. The module $\mathcal{C}\mathrm{smbl}(\mathcal{F};\mathcal{F})$ can be identified with the module of polynomials with coefficients in $\mathcal{F}$, $\mathcal{C}\mathrm{smbl}(\mathcal{F};\mathcal{F}) = \mathcal{F}[\theta_1,\dots,\theta_{n-1},\theta_t]$ and $\mathcal{C}\mathrm{smbl}_x(\mathcal{F};\mathcal{F}) = \mathcal{F}[\theta_1,\dots,\theta_{n-1}]$.

The composition of differential operators in $\mathcal{C}\mathrm{Diff}(\mathcal{F};\mathcal{F})$ induces a multiplication in $\mathcal{C}\mathrm{smbl}(\mathcal{F};\mathcal{F})$, which can be identified with the multiplication of polynomials in the algebra $\mathcal{F}[\theta_1,\dots,\theta_{n-1},\theta_t]$.

In the same way define a module $\mathcal{C}\mathrm{smbl}(Q_1,\dots,Q_p;R) = \mathcal{C}\mathrm{smbl}(Q_1;\mathcal{F})\otimes \cdots \otimes \mathcal{C}\mathrm{smbl}(Q_p;\mathcal{F}) \otimes R$ and for each multilinear $\mathcal{C}$-differential operator the order and the $\mathcal{C}$-symbol by following

$$\mathrm{ord}(\Delta_1 \otimes \cdots \otimes \Delta_p \otimes r) = \mathrm{ord}\,\Delta_1 + \cdots + \mathrm{ord}\,\Delta_p,$$
$$s(\Delta_1 \otimes \cdots \otimes \Delta_p \otimes r) = s(\Delta_1) \otimes \cdots \otimes s(\Delta_p) \otimes r,$$

where $\Delta_i \in \mathcal{C}\mathrm{Diff}(Q_i;\mathcal{F})$, $1 \le i \le p$, $r \in R$.

We identify

$$\mathcal{C}\mathrm{smbl}(Q_1,\dots,Q_p;R) = \mathrm{Hom}_{\mathcal{F}}(Q_1,\dots,Q_p;R) \otimes \mathcal{C}\mathrm{smbl}(\mathcal{F};\mathcal{F}) \otimes \cdots \otimes \mathcal{C}\mathrm{smbl}(\mathcal{F};\mathcal{F}),$$

$$\begin{aligned}&\mathcal{C}\mathrm{smbl}_x(Q_1,\dots,Q_p;R) = \\ &\quad \mathrm{Hom}_{\mathcal{F}}(Q_1,\dots,Q_p;R) \otimes \mathcal{C}\mathrm{smbl}_x(\mathcal{F};\mathcal{F}) \otimes \cdots \otimes \mathcal{C}\mathrm{smbl}_x(\mathcal{F};\mathcal{F}) = \\ &\qquad \mathrm{Hom}_{\mathcal{F}}(Q_1,\dots,Q_p;R) \otimes \mathcal{F}[\theta^j_i],\ 1 \le i \le n-1, 1 \le j \le p.\end{aligned}$$

Let us simplify notations by introducing $\theta = (\theta_1,\dots,\theta_{n-1})$ and $\theta^j = (\theta^j_1,\dots,\theta^j_{n-1})$. The composition of differential operators induces a left $\mathcal{F}[\theta]$-module structure in $\mathcal{C}\mathrm{smbl}(\mathcal{F},\dots,\mathcal{F};\mathcal{F}) = \mathcal{F}[\theta^j_i]$ as follows

$$\Delta_1(\theta) \circ \Delta_2(\theta^1,\dots,\theta^p) = \Delta_1(\theta^1 + \cdots + \theta^p) \cdot \Delta_2(\theta^1,\dots,\theta^p),$$

where $\Delta_1 \cdot \Delta_2$ denotes the multiplication of polynomials.

Theorem 3. *Let $\mathcal{Y}$ be an evolution equation such that the $\mathcal{C}$-symbol of the $\mathcal{C}$-differential operator L_f is nondegenerate on the dense open subset of $\mathcal{Y}$ and* $\mathrm{ord}\,L_f \ge 2$. *Then*

$$E^{p,n-1}_1 = 0,\ p \ge 3.$$

Proof. Let $\Delta \in \mathcal{C}\mathrm{Diff}_{x,p-1}(\mathcal{F}^m;\mathcal{F}^m)$ be a solution of (12) from Theorem 2. Choose a point $\pi \in \mathcal{Y}$ such that the $\mathcal{C}$-symbol $s(L_f)$ is nondegenerate at this point. Denote the $\mathcal{C}$-symbols of Δ and L_f at the point π by $\delta = s(\Delta)(\pi)$ and $\lambda = s(L_f)(\pi)$ respectively. Then $\lambda \in \mathrm{Hom}_{\mathbb{R}}(V;V) \otimes \mathcal{F}[\theta]$, $\delta \in \mathrm{Hom}_{\mathbb{R}}(\bigwedge^{p-1} V;V) \otimes \mathcal{F}[\theta^1,\dots,\theta^{p-1}]$, where $V = \mathbb{R}^m$.

Fix a basis of V and identify λ with an $m \times m$ matrix $\lambda = (\lambda^i_j)$, $i,j = 1,\dots,m$, where $\lambda^i_j = \lambda^i_j(\theta) \in \mathbb{R}[\theta]$ is a homogeneous polynomial of degree $l = \mathrm{ord}\,L_f$. Then $(\lambda^*)^i_j = (-1)^l \lambda^j_i$. Let $\delta^k_{i_1\dots i_{p-1}} = \delta^k_{i_1\dots i_{p-1}}(\theta^1,\dots,\theta^{p-1})$ be the components of δ in the chosen basis. Now one can rewrite the $\mathcal{C}$-symbol of equation (12) from Theorem 2 at point π in the form

$$(-1)^l \sum_{i=1}^{m} \lambda_i^k(\theta^1 + \cdots + \theta^{p-1})\delta^i_{i_1\ldots i_{p-1}} + \sum_{s=1}^{p-1}\sum_{i=1}^{m} \delta^k_{i_1\ldots i_{s-1} i \ldots i_{p-1}} \lambda_i^{i_s}(\theta^s) = 0, \tag{13}$$

where $1 \le i_1, \ldots, i_s, k \le m$.

System (13) can be considered as a linear system of algebraic equations with polynomial coefficients over $\mathbb{C}$. Let us show that the determinant of this system does not equal to 0. Since $\lambda = \lambda(\theta)$ is not degenerate, there exists $v \in \mathbb{C}^m$ such that $\det\lambda(v) \ne 0$. One can suppose that $\lambda(v)$ has an uppertriangular (Jordan) form, $\lambda^i_j(v) = 0$ if $i \ge j$ and $\lambda^i_i(v) \ne 0$. Then for any $\alpha \in \mathbb{C}$ the matrix $\lambda(\alpha v)$ has also an uppertriangular form $\lambda^i_j(\alpha v) = \alpha^l \lambda^i_j(v)$. Since $l = \operatorname{ord} L_f \ge 2$ and $p \ge 3$, there exist complex numbers $\alpha_s \in \mathbb{C}$, $s = 1, \ldots, p-1$, such that for any $i_1, \ldots, i_{p-1}, k$, $1 \le i_1, \ldots, i_{p-1}, k \le m$

$$A_{k,i_1,\ldots,i_{p-1}} = (-1)^l \lambda^k_k(v)(\alpha_1 + \cdots + \alpha_{p-1})^l + \sum_{s=1}^{p-1} \lambda^{i_s}_{i_s}(\alpha_s)^l \ne 0.$$

Put $\theta^i = \alpha_i v$. Then system (13) can be rewritten in an uppertriangular form with diagonal entries $A_{k,i_1,\ldots,i_{p-1}} \ne 0$. Hence, the determinant of system (13) does not equal to 0, and the solution of this system $\delta = 0$. Therefore, $\mathcal{C}$-symbol of Δ vanishes on a dense subset of $\mathcal{Y}$. But this means that $\Delta = 0$. □

4.2. **Scalar evolution equations.** In this subsection we consider the case when $\mathcal{Y}$ is a scalar evolution equation. If the order of an equation $\mathcal{Y}$ is greater then 1, then from Theorem 3 it follows that $E_1^{p,n-1} = 0$ if $p \ge 3$. If the order of $\mathcal{Y}$ is equal to 1 then the following Theorem is true.

Theorem 4. *Let $\mathcal{Y}$ be a scalar evolution equation of the* 1*-st order. Then $E_1^{p,n-1}$, $p \ge 1$, is an infinite dimensional vector space.*

Proof. Locally each scalar evolution equation of the 1-st order by a contact transformation can be reduced to the equation $u_t = 0$. In this case equation (12) has a simple form $D_t(\Delta) = \frac{\partial \Delta}{\partial t} = 0$. Denote by $\mathcal{F}_x$ the algebra of functions on $\mathcal{Y}$ that do not depend on the variable t. Then $E_1^{1,n-1} = \mathcal{F}_x$ and $E_1^{p,n-1}$, $p \ge 2$, consists of operators $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{alt}}_{x,p-1}(\mathcal{F};\mathcal{F})$ with coefficients in $\mathcal{F}_x$ such that $\Delta^{*_s} = -\Delta$, $1 \le s \le p-1$. □

The term $E_1^{p,n-1}$, $p = 1, 2$ need not be trivial. $E_1^{1,n-1}$ contains generating functions of conservation laws. In [3] the term $E_1^{2,1}$ is computed for some evolution equations with single space variable, $n = 2$. Here we compute the term $E_1^{2,n-1}$ for a scalar linear evolution equation with constant coefficients.

Let $\mathcal{Y}$ be a scalar linear evolution equation with constant coefficients $u_t = Lu$, $L \in \mathbb{R}[\theta]$. In this case $L_f = \overline{L}$ is the lifting of the linear differential operator L. We identify L and L_f in the polynomial ring $\mathbb{R}[\theta]$.

The equation defining $E_1^{2,n-1}$ has the following form:

$$D_t(\Delta) + L^* \circ \Delta + \Delta \circ L = 0 \tag{14}$$

and the composition of differential operators $\circ : \mathcal{F}[\theta] \times \mathcal{F}[\theta] \to \mathcal{F}[\theta]$ can be described as follows. Let $\Delta_i = \Delta_i(\theta_1, \ldots, \theta_{n-1}) \in \mathcal{F}[\theta]$, $i = 1, 2$, be polynomials with coefficients in $\mathcal{F}$. Then

$$\begin{aligned}\Delta_1 \circ \Delta_2 &= \Delta_1(\theta_1 + D_1, \ldots, \theta_{n-1} + D_{n-1}) \cdot \Delta_2 \\ &= \Delta_1 \cdot \Delta_2 + \sum_i \frac{\partial \Delta_1}{\partial \theta_i} \cdot D_i(\Delta_2) + \frac{1}{2} \sum_{i \le j} \frac{\partial^2 \Delta_1}{\partial \theta_i \partial \theta_j} \cdot D_{i,j}(\Delta_2) + \ldots,\end{aligned}$$

where $\Delta_1 \cdot \Delta_2$ denotes the multiplication in the polynomial ring $\mathcal{F}[\theta]$ and $D_i \cdot \Delta = D_i(\Delta)$.

If L is a linear $\mathcal{C}$-differential operator with constant coefficients, $L \in \mathbb{R}[\theta]$, then $L = \sum_{i=0}^{k} L_i$, where L_i is a homogeneous polynomial, $\operatorname{ord} L_i = i$, and

$$\Delta \circ L = \Delta \cdot L = L \cdot \Delta, \quad L^* = L(-\theta) = \sum_{i=0}^{k} (-1)^i L_i,$$

where $\Delta \in \mathcal{F}[\theta]$.

Each operator $\Delta \in \mathcal{F}[\theta]$ is a sum of homogeneous operators, $\Delta = \sum_{i=0}^{l} \Delta_i$, $\operatorname{ord} \Delta_i = i$, $\Delta_l \neq 0$. The left-hand side of equation (14) is also a sum of homogeneous operators. Hence one can solve (14) by equating homogeneous terms to 0. Let us wright out 2 terms of maximal order $k + l$ and $k + l - 1$ if $k \geq 2$

$$(1 + (-1)^k) L_k \cdot \Delta_l = 0; \tag{15}$$

$$(1 + (-1)^k) L_k \cdot \Delta_{l-1} + (1 - (-1)^k) L_{k-1} \cdot \Delta_l + \sum_{i=1}^{n-1} \frac{\partial L_k}{\partial \theta_i} \cdot D_i(\Delta_l) = 0. \tag{16}$$

Let $L \in \mathbb{R}[\theta] \subset \mathcal{F}[\theta]$ be a linear $\mathcal{C}$-differential operator with constant coefficients. Consider a transformation

$$L \mapsto L' = e^{-(\lambda,x)} \circ L \circ e^{(\lambda,x)} + A, \quad L' = L(\theta + \lambda) + A, \tag{17}$$

where $\lambda \in \mathbb{R}^{n-1}$, $(\lambda, x) = \lambda_1 x_1 + \cdots + \lambda_{n-1} x_{n-1}$, $A \in \mathbb{R}[\theta]$ is a linear $\mathcal{C}$-differential operator with constant coefficients, $\operatorname{ord} A \leq 1$. Obviously, the operator L' is also a $\mathcal{C}$-differential operator with constant coefficients. We say that $\mathcal{C}$-differential operators L_1 and L_2 are *equivalent* if by applying transformation (17) and a linear change of variables $\theta = (\theta_1, \ldots, \theta_{n-1})$ one can obtain L_2 from L_1. If operators L_1 and L_2

are equivalent then the corresponding evolution equations $u_t = L_1 u$ and $v_t = L_2 v$ can be obtained one from another by a change of variables.

An operator $L \in \mathbb{R}[\theta]$, $L = \sum_{i=0}^{k} L_i$ is said to be *in a normal form* if there exist integer numbers $s_0, \dots, s_k$, $1 \le s_k \le \dots \le s_0 \le n-1$, such that for any $i, 0 \le i \le k$ the following conditions hold:

1. $\frac{\partial L_i}{\partial \theta_j} = 0$, if $j > s_i$;
2. polynomials $\frac{\partial L_i}{\partial \theta_j}$, $s_{i+1} \le j \le s_i$ are linearly independent.

The integer number s_0 is called *the rank* of the operator L and is denoted by $\operatorname{rk} L$.

Lemma 6. *Let $L = \sum_{i=0}^{k} L_i$ be an operator, $L \in \mathbb{R}[\theta_1, \dots, \theta_{n-1}]$. There exists a linear change of variables $(\theta_1, \dots, \theta_{n-1})$ that transforms L to an operator L' in a normal form.*

Proof. Straightforward. □

Theorem 5. *Let $\mathcal{Y}$ be a scalar linear evolution equation with constant coefficients $u_t = Lu$, $L \in \mathbb{R}[\theta]$. If $E_1^{2,n-1} \neq 0$, then L is equivalent to a skew-selfadjoint operator L', $L'^* = -L'$.*

Proof. Let Δ be a $\mathcal{C}$-differential operator satisfying equation (14), $\Delta = \sum_{i=0}^{l} \Delta_i$, $\operatorname{ord} \Delta_i = i$, $\Delta_r \neq 0$. Consider a homogeneous part (15) of equation (14) of the order $r+k$. Since there exists a non trivial solution Δ_l, the order k of L is odd.

By induction, suppose that there exists $r \ge 0$ such that $L_{k-2i+1} = 0$, $0 \le i \le r$. Prove that the operator L is equivalent to an operator $L' = \sum_{i=0}^{k} L'_i$ such that $L'_{k-2i+1} = 0$, $0 \le i \le r+1$.

If $k - 2r - 1 = 0$ then put $L' = L - L_0$. Suppose that $k - 2r - 1 \ge 2$. By Lemma 6 we can assume that there exist integer numbers $s_1, \dots, s_r$, $1 \le s_1 \le \dots \le s_r \le n-1$, such that for any i, $0 \le i \le r$,

1. $\frac{\partial L_{k-2i}}{\partial \theta_j} = 0$, if $j > s_{i+1}$;
2. polynomials $\frac{\partial L_{k-2i}}{\partial \theta_j}$, $s_i \le j \le s_{i+1}$ are linearly independent.

Hence, the homogeneous part of equation (14) of the order $k + l - 1$ (16) simplifies as follows

$$\sum_{i=1}^{s_1} \frac{\partial L_k}{\partial \theta_i} D_i(\Delta_l) = 0.$$

From condition (2) we have that $D_i(\Delta_l) = 0$, $1 \le i \le s_1$. Hence, we proved that if Δ is a solution of equation (14), then $\operatorname{ord} D_i(\Delta) \le \operatorname{ord} \Delta - 1 = l-1$, $1 \le i \le s_1$. But $D_i(\Delta)$ is also a solution of (14) for any i. Therefore, $\operatorname{ord} D_{ij}(\Delta) \le l-2$, $1 \le i, j \le s_1$, and $\operatorname{ord} D_{i_1 \dots i_{l+1}}(\Delta) = 0$, $1 \le i_1, \dots, i_{l+1} \le s_1$. Now it is easy to see that one can find a non zero solution of (14) $\Delta' = D_{i_1 \dots i_j}(\Delta)$ such that $\Delta' \neq 0$ and $D_i(\Delta') = 0$ for any $i = 1, \dots, s_1$. Then

$$L_k^* \circ \Delta' + \Delta' \circ L_k = -L_k(\theta_1 + D_1, \dots, \theta_{s_1} + D_{s_1}) \cdot \Delta' + L_k \cdot \Delta' \\ = -L_k \cdot \Delta' + L_k \cdot \Delta' = 0,$$

and one can continue solving equation (14) with $L = L_{k-2} + \dots$. Repeating arguments above we find a non zero solution of (14) Δ'' such that $D_i(\Delta'') = 0$, $1 \le i \le s_r$. Consider than the homogeneous part of (14) of the order $k + l - 1 - 2r$, which can be written as follows

$$2L_{k-2r-1} \cdot \Delta''_l + \sum_{i=s_r+1}^{n-1} \frac{\partial L_{k-2r}}{\partial \theta_i} \cdot D_i(\Delta''_l) = 0.$$

Since $\Delta''_l \neq 0$,one can find $\lambda_i \in \mathbb{R}$, $s_r + 1 \le i \le n - 1$, such that

$$L_{k-2r-1} + \sum_{i=s_r+1}^{n-1} \frac{\partial L_{k-2r}}{\partial \theta_i} \lambda_i = 0.$$

Put $L' = e^{-(\lambda,x)} \circ L' \circ e^{(\lambda,x)} = L(\theta + \lambda)$, where $(\lambda, x) = \sum_{i=s_r+1}^{n-1} \lambda_i x_i$. Then $L'_i = L_i$ if $k - 2r \le i \le k$ and $L'_{k-2r-1} = 0$.

Therefore, one can find an operator L' equivalent to L such that $L'_i = 0$ for even i. But then we have

$$L'^*(\theta) = L'(-\theta) = -L'$$

and L' is a skew-selfadjoint operator. The Theorem is proved. □

Theorem 6. *Let $L \in \mathbb{R}[\theta]$, $L = \sum_{i=0}^{k} L_i$, be a skew-selfadjoint operator in a normal form and $L_0 = L_1 = 0$. Then*

1. *solutions of equation* (14) *form an algebra $\mathcal{A}$ with respect to the operation of composition;*
2. *if $\Delta \in \mathcal{A}$ then $\Delta^* \in \mathcal{A}$;*
3. *the algebra $\mathcal{A}$ is isomorphic to the algebra of differential operators $\mathcal{A}_0 = \mathbb{R}[X_1, \dots, X_s] \otimes C^\infty(x_{s+1}, \dots, x_{n-1}) \otimes \mathbb{R}[\theta_1, \dots, \theta_{n-1}]$, where $s = \operatorname{rk} L$, and $C^\infty(x_{s+1}, \dots, x_{n-1})$ denotes the algebra of smooth functions depending on variables $x_{s+1}, \dots, x_{n-1}$.*

Proof. Since $L^* = -L$, equation (14) has the following form

$$D_t(\Delta) + [\Delta, L] = 0. \tag{18}$$

If Δ_1, Δ_2 are solutions of (18), then

$$D_t(\Delta_1 \circ \Delta_2) + [\Delta_1 \circ \Delta_2, L] \\ = (D_t(\Delta_1) + [\Delta_1, L]) \circ \Delta_2 + \Delta_1 \circ (D_t(\Delta_2) + [\Delta_2, L]) = 0,$$

and this proves statement (1).

Further, if Δ is a solution of (18), then

$$0 = (D_t(\Delta) + [\Delta, L])^* = D_t(\Delta^*) + [L^*, \Delta^*] = D_t(\Delta^*) + [\Delta^*, L],$$

and (2) is proved.

Now we show that the algebra of solutions of equation (18) $\mathcal{A}$ contains as a subalgebra $\mathcal{A}_0 = \mathbb{R}[X_1, \dots, X_s] \otimes C^\infty(x_{s+1}, \dots, x_{n-1}) \otimes \mathbb{R}[\theta]$. Obviously, each $\Delta \in C^\infty(x_{s+1}, \dots, x_{n-1}) \otimes \mathbb{R}[\theta]$ satisfies equation (18). Let $X_i = x_i + t\frac{\partial L}{\partial \theta_i}$, $1 \le i \le s$. Then

$$D_t(X_i) + [X_i, L] = \frac{\partial L}{\partial \theta_i} + [x_i, L] = \frac{\partial L}{\partial \theta_i} - \frac{\partial L}{\partial \theta_i} = 0,$$

and $X_i \in \mathcal{A}$.

Further,

$$[X_i, X_j] = t[\frac{\partial L}{\partial \theta_i}, x_j] + t[x_i, \frac{\partial L}{\partial \theta_j}] = t\frac{\partial^2 L}{\partial \theta_i \partial \theta_j} - t\frac{\partial^2 L}{\partial \theta_i \partial \theta_j} = 0,$$
$$[X_i, \theta_j] = \delta_{ij}, \quad X_i^* = X_i.$$

Hence, the algebra generated by X_i, $1 \le i \le s$, and $\Delta \in C^\infty(x_{s+1}, \dots, x_{n-1}) \otimes \mathbb{R}[\theta]$ is the algebra of differential operators $\mathcal{A}_0$.

Let us prove that $\mathcal{A} \subset \mathcal{A}_0$. By induction, suppose that if $\Delta \in \mathcal{A}$ and $D_i(\Delta) = 0$ for all i, $1 \le i \le r \le s$, then $\Delta \in \mathcal{A}_0$. Let $\Delta \in \mathcal{A}$ and $D_i(\Delta) = 0$ for all i, $1 \le i \le r-1$. In the same way as in the proof of Theorem 5 one can show that $D_{r-1}^{l+1}(\Delta) = 0$. Hence, Δ is a polynomial with respect to the variable x_{r-1}, $\Delta = \sum_{i=0}^d x_{r-1}^i \Delta^{(i)}$, $D_j(\Delta^{(i)}) = 0$, $1 \le j \le r$. But $\Delta^{(d)} = \frac{1}{d!} D_{r-1}^d(\Delta)$ is a solution of (18) and by the inductive hypothesis $\Delta^{(d)} \in \mathcal{A}_0$. We can apply the same argument to the operator $\Delta' = \Delta - X_{r-1}\Delta^{(d)}$ and show that $\Delta = \sum_{i=0}^d X_{r-1}^i \nabla^{(i)}$, where $\nabla^{(i)} \in \mathcal{A}_0$. Therefore, $\Delta \in \mathcal{A}_0$ and $\mathcal{A} = \mathcal{A}_0$. □

Corollary 3. *Let $\mathcal{Y} = \{u_t = Lu\}$ be a scalar evolution equation, where L is a linear differential operator with constant coefficients. Then term $E_1^{2,n-1}$ of the $\mathcal{C}$-spectral sequence for $\mathcal{Y}$ is isomorphic to a linear space of differential operators $\Delta \in \mathbb{R}[X_1, \dots, X_s] \otimes C^\infty(x_{s+1}, \dots, x_{n-1}) \otimes \mathbb{R}[\theta_1, \dots, \theta_{n-1}]$, $s = \operatorname{rk} L$, such that $\Delta^* = -\Delta$.*

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